At the close of the final session a formal and unanimous resolution was passed expressing appreciation of the hospitality of Columbia University, of the services of Professor Fiske as chairman of the committee on arrangements, and of Professor Hawkes as chairman of the program committee.

President Slaught presided at each of the three sessions. The following seven papers were read:

(1) "Mathematics for the physiologist and physician" by Dr. H. B. WILLIAMS, Assistant Professor of Physiology, College of Physicians and Surgeons.

(2) "The regular solids and the types of crystal symmetry" by Dr. P. L. SAUREL, Professor of Mathematics, College of the City of New York.

(3) "The mathematics of physical chemistry" by Professor G. B. PEGRAM, Dean of the School of Mines, Engineering, and Chemistry, Columbia University.

(4) "The mathematics of biometry" by Dr. L. J. REED, Associate Professor of Biometry and Vital Statistics, Johns Hopkins University.

(5) "An experiment in conducting freshman mathematics courses" by Dr. F. B. WILEY, Professor of Mathematics, Denison University

(6) Preliminary report of the National Committee on Mathematical Requirements by Dr. J. W. YOUNG, Professor of Mathematics, Dartmouth College.

(7) "Mathematics for students of physics" by Dr. LEIGH PAGE, Assistant Professor of Physics, Yale University.

Abstracts of the papers and discussions follow below, the numbers corresponding to the numbers in the list of titles:

(1) It seems to have been generally assumed that students intending to enter upon the study of medicine or the biological sciences need only the most elementary training in mathematics. The biological sciences in the main, however, rest entirely upon the fundamental sciences of physics and chemistry. Without adequate preparation in these sciences the biological student will be greatly handicapped in his studies and if he attempts in later life to enter upon serious original research in his chosen field he will meet almost insurmountable obstacles and be apt to commit most absurd mistakes.

The ideal preparation would be a very thorough training in physics, chemistry, and physical chemistry, with all the mathematical preparation which such training presupposes. Since this would be in itself the work of a life-time, some sort of compromise must be effected. A fairly definite list of the desirable courses in mathematics was given by Dr. Williams.

It was suggested that for students of medicine especially the usual courses in general biology, zoölogy, histology, comparative anatomy and the like be omitted from the preparatory course and mathematics, chemistry and physics be substituted. Given a good ground work in these fundamental sciences, the student will be able at any later time to extend his field of knowledge by independent reading without the help of instructors. Much of the work ordinarily taken in the academic courses by these students is duplicated in the medical school.

Medicine is both science and art. For those who aspire to know anything of the science, training in fundamentals is essential, and if one intends to participate in the advance of the medical sciences a very thorough training should be secured. The best work in physiology has always been done by men whose preliminary training was exceptionally good.

In the discussion of this paper, Professor Slaught told of a student holding the doctor's degree in zoölogy who is now taking a year's course in calculus because it was found necessary for later biological research.

Professor D. E. Smith gave voice to the general feeling of surprise on the part of the hearers that so extensive a training in mathematics was called for. In answer to his questions Dr. Williams stated (1) that students can ordinarily assimilate the material in plane and solid geometry most readily in their high-school years, can then best spare the time for that work and can at that period most easily get into their memory the facts of elementary mathematics; (2) that analytic trigonometry is a useful possession if a student has the time for its study, the speaker himself having used it in science related to medicine; and (3) that the hyperbolic functions are useful for the physiologist in alternating current theory, which is needful in certain researches in physiology.

In answer to inquiries made by Professor Hawkes, Dr. Williams gave further detailed suggestions. The theory of probability and of the errors of observation are important in their physiological bearings. Again, in the study of the nature of nerve impulse the simple physical theory proved inadequate to cover the facts and it was necessary to attack the investigation as a problem in electrical conduction on the theory that the nerve behaves like a submarine cable; measurements were made sufficient to compute the characteristics of the nerve and an artificial cable was constructed such that the curve of electrical response of the artificial cable could hardly be distinguished from that of the nerve studied. It was further found that the variations due to temperature changes agreed with the electrical theory. The studies on the heart by electrical means made by Eintoven of Leyden were possible because of his being well trained in mathematical investigation; his invention of an electrometer which is a well-known modification of the D'Arsonval galvanometer was directed by his study of mathetheory. Many times the physiologist meets with formidable mathematical notation in his reading of physics papers, and he should be able to command this as well as the purely physical material.

Professor Archibald drew attention to O. Fischer's article "Physiologische Mechanik (Bewegungsphysiologie)," 1904, with its bibliography of over four hundred titles, in *Encyclopädie der mathematischen Wissenschaften.*

Professor Jackson reported that Dr. Dryer of Minnesota in a recent address on the use of mathematics in medicine instanced the application of statistics to the study of blood pressure; when the blood volume in animals was compared with bodily weight, it was found that the former was proportional, not to the latter as formerly supposed, but more nearly to the bodily surface, this surface having been found experimentally to be approximately proportional to the two-

thirds power of the weight. There was no suggestion of any a priori reason for the relation between surface and weight such as would have been familiar to one trained in mathematics.

Professor C. N. Moore spoke of the curve representing the time-rate of healing of wounds, which follows an exponential law. Two observations made only a short time apart determine the curve and any important deviations from this curve occurring within the next few days indicate a pathological condition which calls for investigation.

Professor Hurwitz remarked that if the value of mathematics for the various sciences is to be emphasized, college teachers must be able to show such advantages not merely for the exceptional investigator, but as well for a considerable proportion of the students of science.

Professor Bennett related a case where the author of a paper at an educational gathering spoke of the area under a curve as representing the number of observations, and the chairman of the meeting remarked that the paper was very interesting but of course one could not expect to understand the mathematical portion! Professor Bennett also mentioned a limited study of periodic functions as desirable early in the course in algebra.

Professor Fiske closed the discussion by stressing the need of teaching more than the bare matter which the majority will probably use; rather is it our duty to continue as much as ever to give all the mathematical instruction which we find possible.

(2) The object of Professor Saurel's paper was threefold. In the first place, it was pointed out that, as the result of the work of Hessel, Bravais, Gadolin, Curie, and Lorentz, it was now possible to present in very simple form the solution of the problem of finding all the finite groups of operations that can be formed by means of rotations about axes passing through a given point and inversions with respect to that point. In the second place, it was shown that the solution of this problem leads directly to the enumeration of the types of crystal symmetry. Finally, a set of symbols and of names for the thirty-two different types of crystal symmetry was suggested which is believed to be simpler than any set in use at present.

Mr. Wintringham told of Jäger's work on crystal symmetry, the symmetry of animal and vegetable forms, including such studies as that of the pineapple.

Professor Archibald brought out the statement from Professor Saurel that Gadolin's contributions in respect of the thirty-two types of crystal symmetry were not essentially new, his axes of inverse symmetry being equivalent to Hessel's earlier treatment, viz., rotation followed by reflection on a plane, but were more in harmony with the modern methods of treatment. The study by the Greeks of the regular solids (fundamental in crystal classification) involved propositions on "golden section," and Professor Archibald pointed out: (1) that Kepler was enthusiastic regarding this "section" (which he termed "divine") and the manner in which it arose in connection with growths and the Fibonacci

series; and (2) that these elements were fundamental in a theory, likely to revolutionize the whole modern theory of design, described by Mr. Jay Hambidge in *Dynamic Symmetry: The Greek Vase*, soon to be issued by the Yale University Press.

(3) Dean Pegram stated that the student who has college calculus such as provided by the customary three-hour course for a year should find little difficulty mathematically with any of the literature in the most recent text books and treatises in undergraduate courses in physical chemistry. It would be of advantage to the student of physical chemistry if his course in calculus included some elementary work in the solution of differential equations. Students looking forward to advanced work in chemistry should certainly not be allowed to make the mistake of failing to take such a course in the calculus, and preliminary to this it needs to be assumed that they have had good training in algebra and analytic geometry. The subject of the mathematics of statistics as applied both in the theory of measurements, and more especially in the kinetic theory of gases, is of interest in chemistry, but the demands in this direction appear not to be large at the present time.

Teachers of mathematics in college can perform good service to their students who are looking forward to work in chemistry by utilizing preferably for illustrative purposes equations and processes of the type that occur frequently in physical chemistry. At this point in his paper Dean Pegram cited about fifteen types of algebraic and differential equations which the undergraduate student of physical chemistry should be able to handle. These are for the most part familiar to all teachers of elementary calculus, yet it is of value to have them grouped together as they occurred in Dean Pegram's paper.

The Mariotte-Guy Lussac law, $pv \doteq RT$, with the modified forms:

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT, \qquad \text{(Van der Waals)}$$

$$\left(p + \frac{a}{T(v + c)^2}\right)(v - b) = RT, \qquad \text{(Clausius)}$$

$$p(v - b) = RTe^{-(A/RTv)}. \qquad \text{(Dieterici)}$$

Adiabatic equation of a perfect gas, $pv^k = C$.

Expressions for saturated vapor:

$$\log p = a + b\alpha^t + c\beta^t, \quad \text{(Biot)}$$

$$\log p = \alpha - \frac{\beta}{T} - \frac{\gamma}{T^2}, \quad \text{(Rankine)}$$

$$\log p = k_1 + k_2 \log T - \frac{k_3}{T}. \quad \text{(Hertz-Nernst)}$$

First law of thermodynamics:

$$du = dq - dw = dq - pdv;$$

second law of thermodynamics:

$$ds = \frac{dq}{T},$$

with the incidental training needed here in dealing with partial differentiation, exact differentials and related topics.

Ordinary and partial differential equations of the following types:
Monomolecular reaction:

$$\frac{dx}{dt} = k(a - x);$$

bimolecular reaction:

$$\frac{dx}{dt} = k(a - x)(b - x);$$

and similar forms involving three or more factors.

Catalyzed reaction:

$$\frac{dx}{dt} = \frac{k(a - x)}{x^{1/2}}.$$

Diffusion:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}.$$

The normal probability curve ($y = ke^{-\gamma^2 x^2}$) in finite or differential form, with a similar formula arising in the kinetic theory of gases:

$$dn = \frac{4n}{a^3\sqrt{\pi}} e^{-v^2/a^2} v^2 dv.$$

The following were mentioned as the chief texts of interest in this study: Lewes's *System of physical chemistry* (three volumes); Nernst's *Theoretical chemistry*, Washburn's *Physical chemistry*, Partington's *Higher mathematics for students of chemistry*, Mellor's *Higher mathematics for students of physics and chemistry*.

While a good course in calculus prepares a student to do work in physical chemistry, advanced students of chemistry in our universities are finding it very desirable and profitable to put much time on advanced courses in physics and in pure mathematics, since the development of chemistry is sure to be in the direction of more elaborate mathematical formulation of its theory.

In the discussion that followed, Dean Pegram stated that the extent to which the teacher of mathematics should go into the interpretation of these equations is conditioned quite largely by a number of circumstances, such as the student's ability and previous training, the instructor's training and inclination, the place of the mathematics courses in the curriculum relative to the courses in physical chemistry. If the mathematician gives to the student a facility in mathematical operations such as are suggested by these equations, he performs a valuable service. The pupil's interest may well be aroused by giving him the significance of the quantities appearing in these equations if this seems feasible.

Professor Hurwitz mentioned a teacher of thermodynamics who demands considerably more than the amount called for by the speaker, *e.g.*, differential geometry (elliptic, hyperbolic and parabolic points), the non-vanishing of the Jacobian as the condition for reversibility of a transformation. Professor Pegram, however, pointed out that his program had been proposed for undergraduate preparation, not necessarily for graduate students, who might well be expected to continue with more advanced courses suited to their particular needs.

In answer to a question by Professor Burgess, the speaker said that while the ability to fit curves to experimental data may wisely be acquired in undergraduate years if time allows, adequate preparation in the fundamental courses must certainly be assured.

Professor Jackson emphasized the fact that we must be very explicit in our instruction in the fundamental courses, commending the usual and evidently needful practice in thermodynamics of indicating not merely the variable with respect to which the partial differentiation is made, but as well the quantity which is held fast.

(4) In this paper Professor Reed attempted to outline the ideal course of mathematical training for students expecting to enter the field of biometry. In brief this mathematical training consists of courses in algebra, trigonometry, analytic geometry and calculus given in much the usual way, together with a course in the modern statistical methods of treating observed data. This latter should include a discussion of probability leading up to the normal curve, a brief discussion of mechanics treating in particular the principles of moments, and a consideration of methods of treating frequency curves, correlation, and "curve fitting." This statistical theory should be developed from the principles of moments rather than by the familiar "least squares" method as the former is more flexible. The above course would give the student in any of the natural sciences the mathematical background necessary for the treatment of the majority of his problems.

When questioned by Professor Hawkes, Dr. Reed stated that the year course in calculus and a semester course in statistics are sufficient as undergraduate preparation, but that further courses should be added in preparation for research.

Professor Fite emphasized the desirability of some limited amount of probability with examples other than those so commonly drawn from games of chance.

In reply to Professor Richardson, the speaker said that he had for several years given with apparent success a course in calculus, three hours per week for a semester, to students of chemistry, devoting a fair amount of time to drill in using Peirce's *Table of Integrals;* on the other hand Professor Richardson maintained that he had not been able to accomplish this in so short a time. Professor Ransom held that this question depends largely on the number of functions which the student needs to be able to handle, and that for the most part the student in his reading along scientific lines needs not so much to set up the differential equations, nor even to integrate the equations, since the author gives the integra-

tion for the cases most commonly met with; rather does he need to be able to understand the language, to know what it is all about. Professor Jackson expressed his agreement with Professor Richardson, said that even after a solid year's course in calculus the student too often fails to get an adequate grasp of the notions of derivative, integral, etc., just as we may in our despair have noted that the last notion the pupil gets in analytic geometry is the connection between the curve and the equation! On the question of brief courses in calculus, integration chiefly by use of tables, etc., Professor Slaught suggested that what is most needed is a certain maturity of judgment and power which can come only through much practice both in translating problems into the language of calculus and in actually carrying out the steps of the work both in differentiation and integration.

(5) Professor Wiley's paper was inserted at the close of the session on Thursday afternoon by unanimous consent. He has been conducting an experiment in handling freshmen by an individual method which contains some unique features. The plan involves: (1) a laboratory arrangement whereby each student works by himself definite assignments which he gets from a head assistant as rapidly as he is able to complete the work satisfactorily; (2) a corps of assistants chosen chiefly from among the best students in the upper classes, who are rewarded partly by the honor conferred upon them, partly by the benefit derived from the practice teaching involved, and partly, in some cases, by tuition exemption; (3) a scheme of credits based upon the completion of certain assignments and the passing of certain general and special tests. Each student has opportunity to confer with the regular instructor as well as with the special assistants, and groups of students working on practically the same assignments are encouraged to confer together and may meet at intervals for joint conference with the instructor.

The paper inspired numerous questions as to the mechanical details and as to the practicability of the scheme, the answers being given for the most part in the above abstract.

(6) Professor Young made a report of progress on behalf of the National Committee on Mathematical Requirements. In addition to news items of interest already published in previous issues of the MONTHLY, announcement was made of the fact that the U. S. Bureau of Education has agreed to publish the reports of the committee. Professor Young also made a plea for coöperation on the part of teachers of mathematics and mentioned several new reports which are in preparation by the committee. A preliminary report on a suggested revision of the work in mathematics of the first two years of the high school is ready for publication and will soon be issued by the Bureau of Education. Copies of this report will be mailed to all members of the Association and of the Society. This report has been discussed by a large number of secondary associations and clubs, and the committee has had the benefit of suggestions from such sources as well as from individuals all over the country. [Compare *1920*, 145–146].

To Dr. Durell's request that the committee would propose as an alternative a plan incorporating a course in algebra as such and a course in geometry as such, since there are many who object to unified courses, Professor Young replied that the report is meant to cover what the committee believes should be the content of the first two years of the high-school course and that the committee does not intend to suggest any preference as between separate and so-called correlated courses.

Professor Dodd expressed his apprehension that where the committee recommends that "more emphasis should be placed on the acquisition of insight and power and less time devoted to acquiring mere facility in the solution of formal exercises," the latter part might unfortunately be observed by teachers the country over without a like endeavor to adopt the first part.

Professor Young also gave a statement of the fundamental considerations on the basis of which it is proposed to revise the present college entrance requirements. Suggestions from any source are desired in this connection. The statement in full follows.

December 30, 1919.

TO THE COUNCIL OF THE MATHEMATICAL ASSOCIATION OF AMERICA:

The National Committee on Mathematical Requirements was requested by your body at your meeting in December, 1918, and by the Council of the American Mathematical Society also in December, 1918, to undertake a reconsideration of the definitions of college entrance requirements as formulated by a Committee of the American Mathematical Society in 1903.

The National Committee is concerned with the study of a desirable reorganization of courses in mathematics in secondary schools and colleges and with proposals for the improvement in the teaching of this subject generally. The subject of college entrance requirements is only one phase of the work for which this committee was organized.

The National Committee felt that it could not profitably attack the problems connected with the revision of college entrance requirements until it was fully acquainted with present tendencies and movements in secondary education relating to mathematics, nor until its own ideas as to desirable changes in secondary school courses in mathematics had assumed a certain degree of definiteness. Accordingly, the consideration of college entrance requirements was postponed. At a meeting of the National Committee on November 1, 1919, however, the problems connected with the desirable mathematical preparation of college students were discussed. We are able, therefore, to make the following preliminary report of progress:

We call attention to the following facts as bearing on the problem under consideration.

(*a*) There is general recognition of the principle that secondary-school courses in mathematics, especially the introductory courses, should be planned so as to be of the greatest possible value to the pupils taking them, whether these pupils ultimately go to college or not. College entrance requirements should not

influence the organization or the teaching of such courses to the impairment of their general value.

(*b*) The growth of the junior high school movement is developing a group of courses in mathematics for the seventh, eighth, and ninth school years which may differ in organization and classification of subject matter from the traditional division into elementary algebra, plane geometry, etc. A proper articulation between secondary school and college may make it desirable, even though it be not necessary, to take such new organization of subject matter into account in formulating college entrance requirements.

The results secured under present entrance requirements are unsatisfactory mainly for two reasons:

(*a*) Too many students on entering college lack a sufficient understanding of mathematical principles and sufficient power in applying them.

(*b*) Too many such students are found to be weak in carrying out the simpler, but fundamentally important, operations of arithmetic and algebra.

In our opinion the fault is not as much with the topics listed in the definitions of the "units" of college entrance requirements as it is with the type of instruction generally given. The latter, moreover, is greatly influenced by the character of the prevailing type of entrance examination.

We believe that the definitions of "units" should be revised by eliminating from the requirements in algebra some of the technically more complicated and relatively less useful topics (for example, the extraction of the square root of a polynomial) and by specifying more definitely the extent to which the study of the remaining topics should be carried; in particular, by definitely limiting the difficulty and complexity of the formal manipulative exercises. This would allow more time for securing understanding of principles, for practice in applying them, and for thorough drill in the simpler fundamental operations, both numerical and algebraic.

As previously indicated, however, improvement in the preparation of students entering college must be sought primarily in the instruction in secondary schools. In general this instruction does not place sufficient emphasis on understanding and insight, and is too exclusively concerned with the development of mere skill or facility in formal manipulation. The colleges bear a considerable share of the responsibility for this condition, in that the prevailing type of entrance examination, which exerts a powerful influence on instruction, has overemphasized the formal exercise. Steps should, therefore, be taken to bring about a change in such examinations. They should be more consciously formulated to test the candidate's understanding of principles and his power of applying them, as well as his ability to perform the necessary simple fundamental operations accurately and expeditiously. Formal exercises should be in the minority, instead of the majority as at present. More emphasis should be given to verbal problems and these should be graded with the utmost care as to difficulty and as to the powers which they are intended to test.

There should, moreover, be added to the definition of the "units" a definite

statement as to the specific abilities which should be developed (such as the ability to understand and use a formula, the ability to interpret an algebraic result in a concrete setting, the ability to analyze and solve concrete problems, etc.).

The following principles have accordingly been tentatively adopted to govern our proposed reconsideration of college entrance requirements in mathematics:

1. The scholarship implied in the present customary requirement of 2½ units should not be lowered.

2. Drill in algebraic manipulation should be limited to those processes and to the degree of complexity necessary for a thorough understanding of principles and required by the probable applications either in common life or in subsequent mathematics.

3. More emphasis should be given to such immediately useful elementary topics as:

(*a*) The understanding and use of the formula.

(*b*) The interpretation of graphic representation and the use of graphic methods.

4. More emphasis should be placed on the acquisition of insight and power and less time devoted to acquiring mere facility in the solution of formal exercises.

5. While specific minimum requirements in separate subjects like algebra, geometry, etc., are still necessary, adequate provision should be made, perhaps through the so-called comprehensive examination, for the pupils in those schools who are developing "general courses" in mathematics where these subjects are not taught in separate courses.

6. College entrance requirements should be stated not only in terms of subjects and topics but also in terms of specific mathematical abilities to be developed.

7. Means should be found to introduce a proper and desirable amount of flexibility into the requirements in order to encourage progress in secondary education through the experimental introduction of new topics and methods.

On the basis of these principles we hope to formulate our final report for submission to the Council not later than its next summer meeting.

Respectfully submitted, for the Committee,

J. W. YOUNG,
Chairman.

(7) The subject of mathematical needs of students in physics, as presented in Dr. Page's paper naturally divides itself into two parts: first, the mathematical needs of the undergraduate student in physics, and second, those of the graduate.

The elements of algebra and trigonometry are essential prerequisites for an undergraduate course in physics. The parts of algebra of importance are the solution of equations of the first and second degree, simultaneous equations of the first degree, logarithms and exponentials, and the binomial theorem; Dr. Page pleaded for more thorough preparation in these topics and greater practice in

their application, even at the expense of those parts of the subject which the undergraduate is likely to have less occasion to use. At the very beginning of his course in physics the student takes up composition and resolution of vectors, and in order to obtain a clear idea of the significance of these directed quantities he must have a good working knowledge of the elements of trigonometry. It is more important that he should be able to distinguish readily sine from cosine when neither side of the angle is horizontal, than that he should be acquainted with a large number of trigonometrical relations which he may never have occasion to use.

Very early in his course in physics, the student takes up derivations which can be explained satisfactorily only by the use of the calculus. It seems unfortunate to the teacher of physics that the college course in general physics is usually given before the student is able to use the calculus; the student's comprehension of the calculus suffers when he is deprived of the excellent opportunity to apply it which is afforded by the study of physics. The tendency to bring into freshman courses in mathematics at least the elements of differentiation and integration was looked upon with favor, and hope was expressed that this movement will permeate down to the last year of high school.

The mathematical preparation of the man who intends to pursue graduate study and engage in research in physics cannot be too thorough. Its value to him is of two kinds: first, it provides him with tools which are essential for his work, and second, it trains him in those habits of logical thought which are exemplified preëminently in mathematical reasoning.

The first requirement of the graduate student in physics is a good course in advanced calculus and ordinary differential equations. Soon after must come courses in partial differential equations, and the study of Legendre's, Bessel's and Laplace's functions, in order to enable him to solve problems in electrostatics and allied subjects. The historical importance of Fourier's series involved in the solution of problems on the flow of heat has been greatly augmented by applications of this form of analysis to the theory of radiation. Some knowledge of the theory of probabilities is essential before courses in statistical mechanics and the kinetic theory of gases are undertaken.

Above all, emphasis was laid on the importance of vector analysis to the student of physics. With few exceptions, all the important quantities with which physics deals are either vectors or the scalar products of vectors. As mathematics is employed by the physicist as a tool in carrying out reasoning in physics, it is important that the notation employed should emphasize the physical rather than the mathematical significance of the logical processes involved. Hence the superiority of vector over scalar analysis. Of the current notations, none is comparable in simplicity and utility to that of Gibbs. One who has been brought up on this form of vector analysis and has made constant use of it cannot help feeling astonished that it has not been universally adopted by physicists.

In discussing this paper Professor Ransom held that while the speakers on

the program maintained that students expecting to go into scientific work need a wide acquaintance with mathematics, it is still to be said that many students do not know early enough what courses they need or even that there are mathematical subjects which serve their special purposes. Here is the opportunity for a survey course of a half year, where the instructor advisedly does the greater part of the work and suggests what will be of later use, a course which is profitable for sophomores, certainly for juniors and seniors. Such a course would help in furnishing a better knowledge of the *language* of mathematics. Professor Page replied that the student would doubtless not know of the requirements for, say, electrodynamics and the electromagnetic theory of light, where some such course as Gibbs's vector analysis is essential; he will, if wise, ascertain from his instructor in physics or from some other well informed person what the necessary preparation is. Professor Coolidge described a survey course at Harvard characterized by the students as "seeing mathematics"; the course was elected for the most part by those who had no adequate preparation in earlier mathematics and proved a disillusionment for those who had hopefully inaugurated the plan. Professor Hurwitz instanced a similar course at Cornell in which the instructor refused to make it an easy course, the result being that its enrollment has been very small. He added that uneducated laymen can today pick up a popular science journal and can from this acquire the essential principles of new methods in non-technical language; nevertheless only the exterior results can thus be transmitted, and it is our distinct duty to contribute as far as may be through mathematics also to the development of a well-informed public.

Professor Hawkes made the very important remark that much of what is sought through survey courses can be done by mathematical clubs. We need merely to refer to the past few volumes of the MONTHLY to see how much of this desirable work has actually been accomplished in the live universities and colleges of America. Dr. Helen Owens said that the junior mathematical club at Cornell considers application of mathematics to other subjects, thus enabling students to get preliminary notions of the mathematical courses needed in other branches. The experimental plan, which thus far has included applications to physics, statistics, actuarial work and chemistry, has proved very interesting.

A good natured controversy between Professors Bowden and Page as to whether Professor Gibbs was a pure mathematician or a member of the department of physics, turned the discussion to the question of a coördination between the two departments. Professor Page said that the pure mathematician does not keep closely enough in touch with the experimentalist and his work is apt to suffer so far as its immediate usefulness to the physicist is concerned.

Professor Woods emphasized the portion of the paper that called for those parts of differential geometry and other advanced mathematics which in recent years have become of the utmost importance to the physicist and chemist, *e.g.*, the Einstein theory. He also thinks it unfortunate to teach physics without calculus. Under a new plan at the Massachusetts Institute of Technology,

trigonometry is an entrance requirement, the freshman begins with calculus, even before he studies analytic geometry; this course is to be followed by a course in the beginnings of analytic geometry for one term, then further work in calculus.

MEETING OF THE COUNCIL OF THE ASSOCIATION.

The following seventy-two persons and two institutions, on applications duly certified, were elected to membership:

To individual membership.

W. C. BARTOL, A.M. (Bucknell); Ph.D. (Adrian). Prof. of math. and astr., Bucknell Univ., Lewisburg, Pa.

ANNETTE BENNETT, A.M. (Columbia). Head of dept. of math., Eureka Coll., Eureka, Ill.

W. N. BIRCHBY, A.M. (Colo. College). Pasadena, Cal.

L. I. BONNEY, A.B. (Bates College). Asst. prof., Middlebury Coll., Middlebury, Vt.

F. E. BRASCH. Asst. reference librarian, John Crerar Library, Chicago, Ill.

W. M. BRODIE, M.E. (Va. Polytech. Inst.); A.M. (Columbia). Prof., Va. Polytech. Inst., Blacksburg, Va.

J. S. BROWN, A.M. (Texas). Prof., Southwest Texas St. Normal College, San Marcos, Tex.

MARGARET BUCHANAN, A.B. (West Va. Univ.). Grad. stud., Bryn Mawr Coll., Bryn Mawr, Pa.

S. E. CROWE, A.B. (Ohio State). Asst. prof., Mich. Agric. Coll., East Lansing, Mich.

W. L. CRUM, Ph.D. (Yale). Instr., Yale Coll., New Haven, Conn.

R. D. DOUGLASS, A.M. (Maine). Instr., Mass. Inst. of Tech., Cambridge, Mass.

W. F. DOWNEY, A.B. (Amherst). In charge Collins Bldg., English High School Annex, Boston, Mass.

G. C. EVANS, Ph.D. (Harvard). Prof., Rice Inst., Houston, Tex.

A. J. FLEISIG, A. B. (St. Procopius). Instr., St. Procopius Coll., Lisle, Ill.

C. D. GARLOUGH, A.M. (Illinois). Prof., Wheaton Coll., Wheaton, Ill.

H. H. GAVER, A.M. (Virginia). Instr., U. S. Naval Acad., Annapolis, Md.

R. E. GILMAN, Ph.D. (Princeton). Asst. prof., Brown Univ., Providence, R. I.

W. W. GORSLINE, B.S. (Chicago). Teacher of collegiate math. and surv., Crane Jun. Coll., Chicago, Ill.

W. C. GRAUSTEIN, Ph.D. (Bonn). Asst. prof., Harvard Univ., Cambridge, Mass.

T. H. GRONWALL, Ph.D. (Upsala); C.E. (Berlin). Math. and dynamics expert, Technical Staff, U. S. Ord., Washington, D. C.

V. G. GROVE, A.M. (Kentucky). Asst. prof., Mich. Ag. Coll., E. Lansing, Mich.

H. E. GUDHEIM, M.E. (Royal Univ. of Tech., Stockholm). Assoc. prof. of graphics, Va. Polytech. Inst., Blacksburg, Va.

HOWARD HARDING, B.M.E. (Michigan). With Rochester Railway and Light Co., Rochester, N. Y.

W. L. HART, Ph.D. (Chicago). Asst. prof., Univ. of Minnesota, Minneapolis, Minn.

J. R. HITT, B.S. (Miss. Coll.). Asst. prof., Mississippi Coll., Clinton, Miss.

YUN HUANG HO, A.B. (Cornell). Grad. stud., Cornell University, Ithaca, N. Y.

J. B. JACKSON, A.M. (Columbia). Asso. prof., Univ. of South Carolina, Columbia, S. C.

J. S. W. JONES, D.Sc. (Washington Coll.). Prof., Washington Coll., Chestertown, Md.

G. B. KING, A.B. (Ark. Cumberland College). Prof., Cumberland Coll., Clarksville, Ark.

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To institutional membership.

New Hampshire College, Durham, N. H.

The College of Wooster, Wooster, Ohio.

The Council appointed the following Associate Editors of the Monthly for 1920:

Albert A. Bennett,	Cuthbert F. Gummer,	Ulysses G. Mitchell,
Edward L. Dodd,	Derrick N. Lehmer,	Elton J. Moulton,
Otto Dunkel,	Henry P. Manning,	David E. Smith,
Benjamin F. Finkel,	Raymond B. McClenon,	Horace S. Uhler.

It was voted that the incoming president of the Association appoint a committee of five members to report at the summer meeting of the Association on the question of its incorporation and on changes in the constitution necessary to carry such action into effect.

A committee of three was appointed to coöperate with a corresponding committee of the American Mathematical Society, if appointed, to consider the best policy with respect to holding meetings in conjunction in the future.

The invitation of Wellesley College for the Association to hold its summer meeting there in 1921 was referred to this committee.

It was voted that the Association hold its 1920 summer meeting in Chicago if suitable arrangements can be made. A formal invitation has since been received from the University of Chicago.

Annual Business Meeting of the Association.

The secretary-treasurer announced the names of those elected to membership since the summer meeting of the Association. He reported also the death during 1919 of the following six charter members:

C. I. Alexander, professor of mathematics, Texas Christian University (September 7);
C. J. Borgmeyer, professor of mathematics, St. Louis University (December 6);
F. E. Chapman, Gulf Coast Military and Naval Academy (October 1);
W. E. Etzel, rev. father, College of St. Thomas (February 3);
James Maclay, professor of mathematics, Columbia University (November 28);
L. G. Weld, director, Free School of Manual Training, Pullman, Ill. (November 28).

The election of officers for the year 1920 was conducted by mail and in person at this meeting, as provided by the constitution. The tellers (Elizabeth B. Cowley and A. S. Gale) appointed by the Council reported the result of the balloting as follows, 333 ballots having been cast:

For President: R. G. D. Richardson, 162 votes; D. E. Smith, 168 votes.
For Vice-Presidents: Helen A. Merrill, 198 votes; R. E. Moritz, 105 votes; H. L. Rietz, 164 votes; E. J. Wilczynski, 185 votes.
For additional members of the Council (to serve until January, 1923): R. D. Carmichael, 181 votes; E. L. Dodd, 106 votes; E. R. Hedrick, 215 votes; L. C. Karpinski, 115 votes; D. N. Lehmer, 119 votes; H. E. Slaught, 212 votes; Oswald Veblen, 169 votes; J. W. Young, 205 votes.

The following were accordingly declared elected:

President, D. E. Smith, Columbia University.
Vice-Presidents: Helen A. Merrill, Wellesley College, and E. J. Wilczynski, University of Chicago.
Additional members of the Council: R. D. Carmichael, University of Illinois; E. R. Hedrick, University of Missouri; H. E. Slaught, University of Chicago; J. W. Young, Dartmouth College.

In accordance with a provision of the constitution of the Association the Council filled two vacancies on its board by the appointment of E. L. Dodd, University of Texas, and Oswald Veblen, Princeton University. These vacancies were caused by the election of two members of the Council to offices in the Association.

The secretary was authorized to address a letter to the Director General of Railroads protesting against the arbitrary decision by which scientific societies were refused classification as educational organizations and thereby denied the privilege of reduced rates to members attending the meetings of such societies.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to December 15, 1919. The report of the auditing committee (MARY E. SINCLAIR, H. E. SLAUGHT, and J. B. COLEMAN, chairman) was then made. The financial statement is printed in full below.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 15, 1919.

RECEIPTS.			EXPENDITURES.		
Balance Dec. 2, 1918		$3,728.11	Publisher's bills		$2,578.50
			President's office		174.77
1918 subscriptions	$ 11.10		Manager's office		25.24
1918 indiv. dues	121.72		Editor-in-chief's office		182.42
1918 instit. dues	12.75		Other editors' postage		18.35
1919 subscriptions	445.95		Committee on Membership		2.20
1919 indiv. dues	2,857.03		Committee on Dictionary		2.00
1919 instit. dues	404.50		Com. on Math. Requirements		58.12
Initiation fees	114.00		Secretary-Treasurer's office:		
Sale copies of MONTHLY	40.82		Postage	$203.50	
Sale reprints	13.71		Bond	5.00	
Advertising	533.32		Office supplies	11.70	
Exchange	.89		Express, telegrams	7.44	
Interest State Savgs. Bk.	84.22		Clerical work	227.75	
Interest Peoples Bk	54.71		Part expense Register	20.00	
Interest Liberty Bonds	33.57		Printing	296.96	
			Chicago meeting	54.19	
			Michigan meeting	27.83	
Total 1919 receipts		$4,728.29	Paid to sections from initiation fees	33.50	
			Safety deposit rental	2.00	
Total assets up to 1920 business		$8,456.40	Pd. copies of MONTHLY	10.59	
					900.46
			Annals subvention		375.00
Total expenditures		4,317.06	Total expenditures		$4,317.06
			Checking account		500.79
Balance to the end of 1919 business		$4,139.34	State Savgs. Bk. Co. account		1,998.99
			Peoples Bkg. Co. account		1,081.29
Received on 1920 business		441.73	Liberty Bond		500.00
			Victory Bond		500.00
Book balance Dec. 15, 1919		$4,581.07	Bank balance Dec. 15, 1919		$4,581.07

When the accounts were closed on December 15, 1919, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1919 the following items:

Bills Receivable.	
Advertising	$75.00
1919 dues unpaid	75.00
	$150.00

Bills Payable. (Either paid in December or estimated.)	
Publisher's bills, Sept.–Dec.	$1,450.00
Register of members	270.00
December *Annals* subvention	75.00
Init. fees due to sections	90.00
President's office	15.00
Manager's office	15.00
Editor-in-chief's office	30.00
Secretary-treasurer's office	130.00
Printing annual ballot, programs, etc.	125.00
Additional postage	50.00
	$2,250.00

If to the balance on 1919 business shown in this report, $4,139.34, there be added the amounts of bills receivable, $150 and there be subtracted the estimated amount of bills payable, $2,250, there results an estimated final balance on 1919 business of approximately $2,040. Of this surplus, about $1,000 was passed over to the Association by the management of the Monthly when the Association was organized, and this fund is kept by the Council of the Association as a reserve fund. The result for the year's finances is favorable, except for the fact faced one year ago but now made real, viz., an increased rate in the cost of printing dating from August 1, 1919, the exact rates not as yet having been announced by the publishers; this with the steadily decreasing amount derived from advertising makes it necessary to exercise all due care with regard to the expenses of the organization.

W. D. Cairns, *Secretary-Treasurer.*

THE NOVEMBER MEETING OF THE ILLINOIS SECTION.

The first program meeting of the Illinois Section of the Mathematical Association of America was held in room 418 Natural History Building, University of Illinois, on November 22, 1919. There were fifty-nine persons present including the following twenty-four members of the Association: Blumberg, Camp, R. D. Carmichael, Coble, Comstock, Crathorne, Emch, Foberg, Garlough, Kempner, Kustermann, Lytle, McNeill, Bessie I. Miller, G. A. Miller, Risley, Rosenbach, G. H. Scott, Shaw, Slaught, Steimley, E. L. Thompson, Townsend, and Wahlin.

The following program was given: (1) Address: H. E. Slaught, President of the Mathematical Association of America. (2) Round Table Discussion of the Topic:—Freshman Mathematics as related to varying admission credits from the high schools. Discussion led by C. E. Comstock, Bradley Polytechnic Institute, M. W. Coultrap, Northwestern College, G. C. Heritage, Crane Junior College. (3) Address: The Training of Mathematics Teachers, E. B. Lytle, University of Illinois. Discussed by Malcolm McNeill, Lake Forest College, Bessie I. Miller,

Rockford College, W. J. Risley, James Millikin University. (4) Address: How Mathematicians Work, Henry Blumberg, University of Illinois. (5) Business Meeting.

President Slaught sketched the history of mathematics teacher organizations in the United States leading up to the need and opportunities for service of this new Illinois Section. He made an appeal for the hearty coöperation of all college teachers of mathematics in Illinois.

Professor Comstock gave in detail the mathematics preparation of freshmen in Bradley Polytechnic Institute and some other Illinois colleges. He recommended better provision for students entering college with only one unit credit in high-school algebra, a more uniform standard for one unit in algebra, more attention to solid geometry in colleges, and less emphasis upon complicated symbolic manipulations. The general discussion of admission credits was lively and full of interest; all seemed to feel that algebra preparation was very unsatisfactory and numerous pleas for less formal work and more emphasis of general principles and methods were made.

By request, Dr. Lytle read his Ann Arbor address on the training of mathematics teachers; this address appears in full in the present number of the MONTHLY. The discussion put emphasis on the point of giving teachers more explicit training in the use of libraries, and on more independent preparation and forceful presentation of reports on special topics; Miss Miller offered a definite scheme for carrying out and grading such work.

Professor Blumberg described numerous methods of work, both humorous and serious, used by certain well-known mathematicians and concluded with a few generalizations on effective methods of work in the field of mathematics.

The old officers were reëlected for one more year; Chairman, J. A. FOBERG, Crane Junior College, Secretary-Treasurer, E. B. LYTLE, University of Illinois, members of Executive Committee, C. E. COMSTOCK, Bradley Polytechnic Institute, G. T. SELLEW, Knox College, L. S. SHIVELY, Mount Morris College. The question of the place and time of the next meeting was discussed and finally referred to the Executive Committee with power.

At the close of the program a group took luncheon together at the Illinois Union building.

ERNEST B. LYTLE, *Secretary-Treasurer.*

1820

First meeting of the Royal Astronomical Society, London, January 12; V. A. Puiseux, French geometer and analyst, born April 16; John Casey, Irish mathematician, born May 12; J. P. E. de Fauque de Jonquières, French mathematician and vice-admiral, born July 3; W. J. Rankine, Scotch engineer, born July 5; J. C. Houzeau de la Haye, Belgian bibliographer of astronomy, born October 7.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

A number of questions have been standing in this department for some time with no replies or with replies which still leave something to be desired in the way of completeness or finality. Several of these are reprinted below, with the intention of directing our readers' interest to them again, and stimulating replies. To those which have already received some attention short notes are appended, indicating the extent to which they still remain open for consideration.

QUESTIONS.

15. In the *Proceedings of the Royal Society of Edinburgh*, vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

A partial reply to the above has been received showing that if $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$. Can some one show how the "easy proof" then follows?

21. For the Diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$\begin{array}{rrrrrrrrr} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given Diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

In a reply to this question published in the MONTHLY for June, 1919, E. B. Escott showed how to obtain an infinite number of rational solutions of the equation, and in particular all the integral solutions contained in the above list. References to the literature given in connection with this reply indicate the existence of an infinite number of integral solutions. Thus one question explicitly raised regarding the equation still awaits an answer.

30. A certain Normal University wishes to offer thirty-five hours of college mathematics for the benefit of high-school teachers. What should these courses be in order that, primarily, they may be of the greatest value to high-school teachers of mathematics and, secondarily, that they may furnish stimulus for a more extended pursuit of the subject?

No reply to this question has been received. Some idea of conditions as they are may be obtained from the replies to another question in the MONTHLY for December, 1916, pp. 395–399.

34. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

In a remark published in the MONTHLY for September, 1917, S. A. Corey showed by reference to a previous paper that the stated relation is satisfied whenever $f(x)$ is a polynomial of degree not exceeding 4. It has not yet been completely proved, however, that this is the only possible form for $f(x)$.

35. Is the following theorem new or has it previously been published?

THEOREM. *If two parallel planes, π and π', cut sections from a cylindrical surface S and two spherical surfaces S_1 and S_2, and if the sum of the sections of S_2 is equal in area to the sum of the sections of S plus the sections of S_1, then the part of S_2 included between π and π' is equal in volume to the sum of the parts of S and S_1 included between π and π'.*

In communicating this question, E. O. Brower commented on the simplicity with which the theorem could be proved, which led him to wonder whether so simple a proposition had remained unnoticed.

37. Criticize the following as fundamental definitions of elementary geometry:

A *plane surface* is the limit approached by a finite portion of the surface of a sphere as the radius increases without limit.

A *straight line* is the limit approached by a finite portion of the circumference of a circle as the radius increases without limit.

38. As the several courses in secondary and collegiate mathematics are now taught there is a noticeable difference of treatment with respect to the relative emphasis placed on logical accuracy and on development of technique. In elementary algebra, technique predominates, while in plane and solid geometry and advanced algebra logic is more strongly stressed. Trigonometry, analytic geometry, and the calculus are less easily classified; but there is at least a general tendency to emphasize logic in analytic geometry and technique in the calculus. It is suggested that a general appraisal of the reasons for this difference in treatment, its value, and possible alterations, would be helpful.

DISCUSSIONS.

It is earnestly desired that the heading "Discussions" in the title of this department shall be something more than a mere name. It is hoped that articles published here will frequently give rise to others on like topics, and that in many cases valuable and stimulating interchanges of views may result. Several papers published as "Discussions" in these columns recently have explicitly requested the opinions of other mathematicians in support or opposition; more than half of the discussions in late numbers are well adapted to bringing out further ideas. Why not read over some of them, write out your opinions, and submit them?

An instance of the sort just referred to is found in the first discussion this month. In November, 1919, Professor Moritz gave a relatively elementary derivation of the formula for the quadrature of the parabola. In the present number, Professor Dunkel presents another way of obtaining the result. Professor Dunkel's work has the merit of extreme simplicity; it also remains in the closest agreement with the definition of "area" and "definite integral" to which we are led in the calculus. On the other hand, the method by which the limit involved in the problem is obtained, while very easy, is distinctly an artifice. A comparison of the two derivations will prove interesting.

In the second discussion Professor Lovitt gives the result of an investigation into the inaccuracies found in the treatment of inverse trigonometric functions in current texts. Every teacher will agree that the statements frequently found

may violate all notion of logical precision. An annoying situation of a type less vicious than the cases enumerated by Professor Lovitt is brought about by a text-book which defines the principal value of an inverse function as the smallest positive value.

In the last discussion Professor Rees shows how the motion of a body acted on only by the force of gravity and a resistance proportional to velocity may be readily and easily studied by the use of differential equations and initial conditions in vector form. This paper affords a good instance of the economy, both in notation and actual work, resulting from the use of single vector relations in place of triplets of scalar relations.

I. Note on the Quadrature of the Parabola.

By Otto Dunkel, Washington University.

The article by Professor Moritz entitled "On the Quadrature of the Parabola" in the November issue of the Monthly has suggested to the writer to present another derivation of the same result, since, in addition to being fairly simple, this second method follows directly the classic process of defining an area as the common limit of an inferior and a superior sum. This development might be found easy enough to serve as an illustrative example in the presentation of the summation formula in the integral calculus.

Let the equation of the curve be $y = x^m$, $m =$ a positive integer, and suppose that it is desired to obtain the area between the curve, the x-axis and the ordinates at $x = a$ and $x = b$, where b is greater than a and both are positive. Divide the interval from a to b on the x-axis into n subintervals, equal or unequal, and upon them as bases erect two sets of rectangles, the one inscribed and the other circumscribed. The sums of the areas of these rectangles are respectively,

$$(1) \qquad I_n = \Sigma x_i^m(x_{i+1} - x_i), \qquad S_n = \Sigma x_{i+1}^m(x_{i+1} - x_i).$$

It will be shown that as n becomes infinite so that the length of the longest subinterval, δ, approaches zero, each of these sums approaches the same limit, which by the usual definition is the area desired. This limit will also be determined in the process.

It may easily be seen from a figure, especially when all the subintervals are equal, that

$$(2) \qquad \text{Limit } (S_n - I_n) = 0.$$

This also follows algebraically, for

$$S_n - I_n = \Sigma(x_{i+1}^m - x_i^m)(x_{i+1} - x_i) \leqq \delta\Sigma(x_{i+1}^m - x_i^m).$$

In the latter summation all the terms cancel except the first and last, and hence the difference, $S_n - I_n$, is less than or equal to $\delta(b^m - a^m)$. Thus it follows that (2) is true.

A quantity will now be determined which lies between I_n and S_n and is independent of n. It is clear that x_{i+1}^m, $x_{i+1}^{m-1}x_i$, $x_{i+1}^{m-2}x_i^2$, $\cdots$, x_i^m form a decreasing sequence of $m+1$ positive terms and hence their arithmetic mean is greater than the smallest term x_i^m. Using this inequality it follows that

$$I_n < \Sigma(x_{i+1} - x_i)\frac{x_{i+1}^m + x_{i+1}^{m-1}x_i + x_{i+1}^{m-2}x_i^2 + \cdots + x_i^m}{m+1},$$

$$< \frac{1}{m+1}\Sigma(x_{i+1}^{m+1} - x_i^{m+1}) = \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1}.$$

In a similar manner an inequality is found for S_n, and hence

$$0 < I_n < \frac{b^{m+1}}{m+1} - \frac{a^{m+1}}{m+1} < S_n. \tag{3}$$

By representing the three quantities I_n, $b^{m+1}/(m+1) - a^{m+1}/(m+1)$, S_n by points on a straight line and by considering the meaning of (2) and (3) as applied to these points, it will be obvious that the common limit of I_n and S_n is $b^{m+1}/(m+1) - a^{m+1}/(m+1)$. This then is the expression for the desired area. If the equation of the curve is $y = px^m$, it will be readily seen that the above result must be multiplied by p.

The same method, with a slight amount of extra manipulation, may be used for negative values of m and also for fractional values, excepting, however, the special case $m = -1$.[1]

An elementary evaluation of the area of any segment of an ordinary parabola by means of special properties of the curve is given in the *Traité de Géométrie* by Rouché et Comberousse, 2d vol., p. 348 (8th ed., 1912). The properties here used are such as might be given in the ordinary text on analytics. A somewhat similar treatment occurs in the first volume of Goursat-Hedrick's *Mathematical Analysis*, p. 134, and is referred to as one of the processes used by Archimedes. Here the summation of a geometric series is employed, but this may be avoided and the proof simplified by comparing the areas of the interior triangles with certain corresponding exterior triangles. These two proofs are somewhat similar to the one employed by Professor Moritz.

II. Inverse Trigonometric Functions.

By W. V. Lovitt, Colorado College.

As I look over the available text-books on trigonometry the feeling grows upon me that they are hastily written and some topics inadequately treated. It is certain that many errors are present.

I have examined twenty-four different texts with special reference to their treatment of the inverse trigonometric functions. In five the treatment was so

[1] An elementary treatment of this case was given by the writer under the title, A Geometric Treatment of the Exponential Function, in *Washington University Studies*, scientific series, vol. 6, no. 2, p. 33.

short (less than a page) that freedom from a positive error might be attributed to shortness. There are three which, although their treatment is also short (about two pages) are entirely correct and give positive warning against some of the errors which some of the others commit. They do not have a single example or exercise which could be criticized. Six more are to be commended in that what they do say in way of exposition is correct. They neglect however to point out a single pitfall and some of their exercises mislead the student. For instance, prove that

$$\arccos (1 - 2m^2) \equiv 2 \arcsin m.$$

They fail to point out that the expressions on opposite sides of the identity symbol do not represent the same sequence of angles; or that if we confine ourselves to the principal values of the angles the identity does not hold for all values of m, e.g. $m = -\frac{1}{2}$. It must be understood that this equation is true only for particular choices of the various possible values of the functions.

The remaining ten have positive errors. To quote from one author without naming him "the principal value is the one to be used in the following examples." Among the examples I find: Prove

$$\arccos (-4/7) = \arctan (-\sqrt{33}/4).$$

Nine authors make this same direct statement that the principal value only is to be used in a given list of problems. From their lists of problems I select five more, no two from the same author

$$2 \arctan 2 = \arcsin 4/5,$$

$$\arcsin 1/2 + \arccos \sqrt{3}/2 + \arctan \sqrt{3} = \arcsin \sqrt{3}/2,$$

$$2 \arctan x = \arccos (1 - x^2)/(1 + x^2) \quad [\text{try } x = 2][1],$$

$$2 \arccos x = \arcsin (2x \sqrt{1 - x^2}) \quad [\text{try } x = 1/2],$$

$$\arctan m/n - \arctan (m - n)/(m + n) = \pi/4 \quad [\text{try } m = 2, n = -1].$$

Six of the nine ask the student to prove that

$$\arctan m + \arctan n = \arctan (m + n)/(1 - mn). \tag{1}$$

But (1) is not true for principal values if $\arctan m + \arctan n$ is less than $-\pi/2$ or greater than $+\pi/2$. It is not true for instance when $m = n = +\sqrt{3}$.

The tenth man, who is cautious, says that "it is customary to limit the values of an inverse circular function to the smallest value." He asks us about a half page later to prove (1) and also to prove

$$\arcsin 3/5 + \arcsin 8/17 = \arcsin 77/85. \tag{2}$$

If the principal values only are to be used then (1) is not always true as has already been pointed out. If other values are allowed (1) is true only for particular choices of the values of the functions involved. The latter remark applies equally

[1] Suggestions in brackets are my own.

well to equation (2). For example, (2) is not true if arcsin 3/5 and arcsin 8/17 are angles whose terminal lines lie in the second quadrant.

Direction is given in some instances to prove the statement (1) by taking the tangent of both sides. Without some restrictions on the angles involved this does not prove the equality; else

$$60° = 240°$$

because it happens that

$$\tan 60° = \tan 240°.$$

Is it any longer a mystery why the student does not obtain a better grasp of his mathematics?

III. The Path of a Projectile when the Resistance Varies as the Velocity.

By E. L. Rees, University of Kentucky.

The very simple problem of finding the path (and its hodograph) of a projectile in a vacuum is treated in most elementary treatises on vector analysis, but the problem of finding the trajectory when the body moves in a resisting medium is usually not considered as the general problem does not lend itself readily to vector treatment. However, there is a special case of some interest which presents no difficulties. If we assume that the resistance varies as the velocity (which under certain conditions is approximately true for low velocities in the air) the vector differential equation of motion is of a very simple type and its solution differs in no essential way from that of a scalar differential equation of the same type. The equation of such a motion is

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{g} - k\frac{d\boldsymbol{r}}{dt},$$

where $\boldsymbol{g}$ is the vector acceleration due to gravity, and k is a positive scalar const. This is a linear vector differential equation which may be integrated at once by applying the integrating factor e^{kt}.

Integrating we get

$$\frac{d\boldsymbol{r}}{dt} = \frac{\boldsymbol{g}}{k} + e^{-kt}\boldsymbol{C}_1$$

and

$$\boldsymbol{r} = \frac{\boldsymbol{g}t}{k} - \frac{e^{-kt}}{k}\boldsymbol{C}_1 + \boldsymbol{C}_2,$$

where $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are the vector constants of integration.

Assuming $\boldsymbol{r} = 0$, $\boldsymbol{v} = \boldsymbol{v}_0$ when $t = 0$ we have

$$\frac{d\boldsymbol{r}}{dt} = \frac{\boldsymbol{g}}{k} + e^{-kt}(\boldsymbol{v}_0 - \boldsymbol{g}/k) \tag{1}$$

and

$$\boldsymbol{r} = \frac{\boldsymbol{g}t}{k} + \frac{1}{k}(1 - e^{-kt})(\boldsymbol{v}_0 - \boldsymbol{g}/k). \tag{2}$$

The first of these equations is the equation of the hodograph and the second is the vector equation of the trajectory.

The highest point reached by the projectile may be found by multiplying the first equation dotwise by $\boldsymbol{g}$ and setting $\boldsymbol{g}\cdot(d\boldsymbol{r}/dt) = 0$. Solving this equation for t we find $t = 1/k \log (1 - k\boldsymbol{v}_0\cdot\boldsymbol{g}^{-1})$, and this substituted in equations (1) and (2) gives the velocity and the position vector of the projectile at the highest point. Similarly, to find the time of flight and the range on the horizontal plane through the point of projection multiply the second equation by $\boldsymbol{g}\cdot$ and set $\boldsymbol{g}\cdot\boldsymbol{r} = 0$, solve for t, etc.

Since k is positive and t varies from zero to infinity e^{-kt} varies from 1 to 0 and the hodograph is seen to be the segment of a line which terminates at the tips of the vectors $\boldsymbol{v}_0$ and $\boldsymbol{g}/k$.

If the initial velocity is allowed to take all possible directions the hodographs will form a bundle of line segments extending from the tip of $\boldsymbol{g}/k$. If the initial speed is constant the segments will terminate in the surface of a sphere with center at the origin and radius v_0. If the angle of projection varies in a vertical plane and the initial speed is constant the segments will form a pencil one end of each segment resting on the tip of $\boldsymbol{g}/k$ (vertex of pencil) and the other on a circle of radius v_0. The vertex of the bundle or pencil corresponds to the limiting velocity which is the same for all initial velocities.

The cartesian equation of the trajectory is found from the vector equation in the usual way to be

$$y = \frac{g}{k^2}\log(1 - x/l) + \frac{mx}{l},$$

where

$$l = \frac{v_0\cos\alpha}{k} \quad \text{and} \quad m = \frac{v_0\sin\alpha}{k} + \frac{g}{k^2},$$

α being the angle of projection.

From this equation it is seen that the trajectory has an asymptote whose equation is $x = l$, which accords with the known fact that the trajectory of a body moving in any resisting medium has a vertical asymptote.

RECENT PUBLICATIONS.

REVIEWS.

CAJORI'S HISTORY OF MATHEMATICS.

A History of Mathematics. By FLORIAN CAJORI. New York, The Macmillan Company, 1919; pp. x + 514. Price $4.

It is twenty-five years since Professor Cajori, then already known for his contributions in the field of mathematical history, published the first edition of the work under review. During this period the book has been reprinted six times, and the author and publisher have now united in giving to students a second edition, to a considerable extent rewritten and in certain chapters materially enlarged.

The general plan of the first edition has been followed, the work being divided into three rather distinct parts, although not in any formal manner. The first part may be described as relating chiefly to the development of elementary mathematics and including the following main topics: the Babylonians, the Egyptians, the Greeks, the Romans, the Maya, the Chinese, the Japanese, the Hindus, the Arabs, and Europe during the Middle Ages. This part occupies 129 pages, or a little more than one fourth of the text. All things considered, this is a fair allotment of space.

The section relating to Europe during the sixteenth, seventeenth, and eighteenth centuries may be considered as the second part and may be described as concerned with the rise of higher mathematics. Naturally a large part of the work of these centuries was elementary, pertaining to the establishing of the present algebraic symbolism, to the discovery of the common theorems relating to equations, and to the better understanding of the nature of series; but the birth of higher mathematics, including modern higher geometry, the group theory, the general notion of functions, and the laying of scientific foundations, may also be said to have taken place in this period. To these centuries the author has assigned 148 pages, or somewhat more than 30 per cent of the text,—again a fair distribution of space.

The rest of the work is devoted to the period from 1800 to the present time, and includes 209 pages, or upwards of 43 per cent of the text. Roughly speaking, the space assigned to each of the three periods is in the ratio of 25 : 31 : 44, and for the purposes of students who will use this book the ratio is both significant and reasonable.

The first part of the work may be described as giving a racial view of the development of mathematics. This is natural and, for practical purposes, is necessary. Commerce had not yet come to the aid of scholars in the free transmission of thought, and racial characteristics predominated in science as they did in art, religion, and modes of government. In treating of Europe in the Middle Ages, however, Professor Cajori abandons the attempt to classify by races, tending apparently, and with some reason, to classify most of the scholars as churchmen and to consider Alcuin, Gerbert, Johannes Hispalensis, and Leonardo Fibonacci as belonging to the same branch of the human family tree.

In the second part of the work the tendency is to classify either by centuries or by periods which may be characterized by the influence of such men as Vieta, Descartes, Newton, and Euler. This is a natural and allowable treatment, although not the classification which some of those who give courses in the subject may prefer.

In the third part of the work the classification is by such subjects as synthetic geometry, analytic geometry, and algebra. Such a classification is, for the instructor in mathematics, the most helpful of all, and many readers will wish that Professor Cajori might have seen his way to extending it so as also to include the mathematical topics studied in the elementary school, the high school, and the junior college.

Taking the work as a whole, the classification of material is justifiable and the amount of information exceeds that to be found in any other general history of mathematics that has thus far appeared in English.

Even though one may read the book with a friendly eye it is probable that he will find himself giving expression to regret from time to time that Professor Cajori should have adopted certain forms of expression or have made certain definite assertions that are open to question. If a critic happens to have the audacity of youth, he will perhaps find himself giving a freedom to his pen that he will not approve in his later years. If he has himself yielded to the lure of authorship, he will remember Cowper's line,

> None but an author knows an author's cares,

and will seek to be more charitable. When we consider that the first volume of Cantor's monumental work appeared forty years ago, and that for the last twenty years there has been maintained in the third series of the *Bibliotheca Mathematica* a department devoted to a severe criticism of that great treatise, we realize how easy it is to find fault with any book. Indeed, it often seems as if it requires more genius on the part of a critic to refrain from petty faultfinding than it does to record a hundred slips of the memory, of the judgment, or of the pen.

The reviewer now proposes to mention certain typical excellencies of Professor Cajori's work, and then to mention certain types of defects, leaving the details of these defects for others to find, or for such helpful criticism as may be acceptable when another edition is under consideration.

Among the features which will render the work helpful to students of the history of mathematics there may first be mentioned the considerable bibliography given in the footnotes. Such assistance is valuable not merely in supporting some particular assertion of the author's but in showing to the student that there is an extensive literature awaiting him in case he cares to pursue the study further. It is not necessary that the books referred to should all be easily accessible, and in this case some of them are not; so long as they are worthy they act as a stimulus to further study, and on this account the list that has been given might profitably have been much longer.

Another feature that seems to this reviewer very fortunate is that of quoting from the original source in various cases in which the language is particularly significant. For this purpose Professor Cajori has assumed on the part of his readers some knowledge of French, Latin, and German, and it will not often be the case that students are so ignorant of at least two of these languages as not to profit by the quotations given. There is, however, a stronger reason for commending the plan, namely, that the student is thereby encouraged to feel that he is prepared to go to the original sources to secure first-hand information. This reviewer would have advised even more, much more, of this style of quotation, placing the material in footnotes so as not to disturb the reader who can not use it, and not hesitating to include easy Italian; but this is merely a matter of personal preference; the significant thing is that the step has been taken.

A third commendable feature is that of revealing the human side of those scholars whose names are mentioned, thus showing the history of mathematics as the history of men. Plutarch was none the less of a historian because he was the world's greatest biographer. Professor Cajori has given a great amount of information about people, and in the main this information has been well chosen.

When it comes to a consideration of types of those features which may be classed with "things one would rather have left unsaid," perhaps the first general criticism in the minds of most readers will relate to the style in which the book is written. "It is style alone," said the elder Disraeli, "by which posterity will judge of a great work, for an author can have nothing truly his own but his style." While this assertion is open to the criticism that applies to most epigrams, no one can read the treatise under review without sympathizing with Voltaire's advice, "In every author let us distinguish the man from his works." Professor Cajori does not talk as he too often writes. His habit of writing so many sentences in a kind of inverted order will probably annoy most of his readers. While often an aid to emphasis, or a desirable variant to one's usual style, or a means of connecting two sentences euphonically, this order is used so frequently by Professor Cajori as to detract from the clearness of the expression and to become an unfortunate mannerism. Thus we have within a few lines of each other, the sentences, "A profound scholar . . . and a mathematician . . . was Pierre de Fermat;" "A great contribution to geometry was his *De maximis et minimis;*" and "A contemporary mathematician . . . was Blaise Pascal." Sometimes this unfortunate habit even leads to temporary confusion, as in the sentence beginning, "About forty years after Archimedes flourished Apollonius . . .," in which the reader must go to the end of the sentence to find which name serves as the subject in the phrase quoted. Occasionally the skein becomes even more tangled, as in the sentence, "Greater taste than for geometry was shown by the Hindus for trigonometry"—a sentence which can easily be parsed, but which is nevertheless unfortunately expressed. Such constructions are an author's own property; he has a legal right to his own style, and a moral right as well. Nevertheless one cannot read the book without frequently being conscious of the remark of Disraeli's quoted above. It is a safe assertion that Professor Cajori himself would be greatly surprised to know the number of times this peculiarity of style asserts itself. Somehow, too, this reviewer does not connect the man who wrote this book with a sentence like the following; "We abstain from introducing additional Greek opinion regarding Egyptian mathematics, *or from indulging in wild conjectures.*" The sentence is perfectly grammatical, but it lacks the dignity which characterizes the author.

Somewhat related to the matter of style is the frequent use of a man's initials when there seems to be nothing gained. This is seen, for example, in the sentence (p. 222) relating to Jean Bernoulli: "Johann admired the merits of G. W. Leibniz and L. Euler, but was blind to those of I. Newton." If there were any other Leibniz, Euler, or Newton whom the reader would naturally have in mind, the use of the initials, while unusual, might be understood; but to see such initials

with about half the names in the book is a matter of some annoyance. If the dates had been so frequently given with the names as to impress them on the mind, or to aid the reader to place certain men in proper chronological relation to others, there would have been a decided gain to counterbalance any annoyance arising from the appearance of the page; but one would as soon refer to W. Shakespeare and D. L. George as to I. Newton. One of the curiosities of this method of reference is seen in the case of P. Mersenne (p. 156) whose first name was Marin (p. 163), the P. evidently standing for Pater or Père.

It may be pardonable to mention also, in connection with the matter of style, the author's habit of referring to a writer whose biography has not already been given, or, indeed, to one who is quite unknown and who is never mentioned again. The student, who reads on page 32 what "H. G. Zeuthen finds," does not learn who H. G. Zeuthen is until he reads page 190; and when he reads (p. 44) the opinion of Venturi, he may search the book in vain to find who Venturi was or what his opinion may be worth. The same may be said of Thymaridas (pp. 59, 111), and a similar criticism may be made with respect to the Palatine Anthology, a work of which the student is quite certain to be ignorant. In this connection it may be suggested that it would have been better had "Thomas Finck, a native of Flensburg" (p. 151) been spoken of with reference to his nationality, since few readers will have the slightest idea in what country Flensburg is situated.

A second point of friendly criticism, for we should all be friendly in recognizing the amount of labor put upon this work, may properly relate to the statements of fact. Here the critic of the Eneström type will find more than this reviewer wishes were possible. A few of these statements will suffice to make clear the assertion that the book is not without its errors. That the Babylonian notation employed only two principles—"the additive and multiplicative" (p. 4) is itself contradicted on the same page in the statement that the subtractive principle was also employed. Indeed it should be said that the evidence of this fact is not confined by any means to the tablets found at Nippur. The frequent assertion that the Babylonians used sexagesimal fractions will be misunderstood; we have no evidence of such fractions before the Greeks introduced them into astronomy, even though the Babylonian use of sixty as a kind of radix in notation is well established, not merely by "two Babylonian tablets" but by many others. The fact that 30 sometimes meant $\frac{30}{60}$ does not illustrate the use of what the reader will understand by sexagesimal fractions, namely, degrees, minutes, seconds, thirds, and so on.

The date of Ahmes is, of course, still a matter of conjecture. When, however, it is given as "some time before 1700 B.C.," and when it next appears in the phrase "the incorrect formulæ of Ahmes of 3000 B.C.," and when it is next referred to as 3,000 years before Leonardo Fibonacci wrote his *Liber abaci* (1202 A.D.), it is safe to say that students will regret that certain assertions are made so positively.

The statement that "to Egypt Greece" is indebted for elementary geometry

(p. 15) is, of course, a matter of definition; but it will be misleading to the average reader until he has read five lines further on the page. The positive statement that Pythagoras sacrificed a hecatomb (p. 2) is not placed in doubt until the student has proceeded sixteen pages further in his reading. The statement that the method of exhaustion infers that the *circle* is exhausted (p. 23) is later corrected (p. 25) by the assertion that it is the "spaces between the polygons and circumferences" that are exhausted.

The statement that Plato did not acquire his taste for mathematics from Socrates (p. 25) is a rather strong one. In Dr. Jowett's delightful paraphrase of the *Theætetus* it is said that "he, Socrates, is a midwife, although this is a secret; he has inherited the art from his mother bold and bluff, and he ushers into light, not children, but the thoughts of men." Who knows how many thoughts of Plato were delivered by Socrates in this quasi-obstetric capacity?

That "Athens produced the greatest scientists and philosophers of antiquity" (p. 29) is a statement that will probably be criticized. As a matter of fact, it is always a cause for surprise that continental Greece produced so few scholars of this type in comparison with the islands of the Ægean Sea and with the Greek colonies.

Speaking of a space of 150 years, the statement that Sextus Julius Africanus was "an occupant of this long gap" sounds rather odd, although it is not probable that anyone will misinterpret it. That Pascal did not lead a "quiet and unaggressive life" (p. 163) will be an unpleasant surprise for most readers, and it would be interesting to know just what extended period of his life Professor Cajori had in mind in making the assertion.

Among the errors of fact there should be mentioned the subject of dates, already referred to in the case of Ahmes. It is not a matter of great importance to most readers whether Thales was born in 640 B.C. and died in 546 B.C., or whether the dates should be c. 640 and c. 542 respectively; and whether the eclipse which he prophesied took place in 585 B.C., or, as some writers assert, twenty-four years earlier. All that the general reader ordinarily cares for is an approximate date. Thus when the dates "580?–500? B.C." are given for Pythagoras, the purposes are sufficiently served. If, however, the dates for Pythagoras are queried, why should not the three dates for Thales also be given as doubtful? This is a work of reference and it is desirable that such statements of fact be accurate and that a doubt be expressed where one is generally supposed by good writers to exist. The same criticism applies in many cases throughout the book, including those relating to Anaxagoras, Archytas, Plato, Bede, Eratosthenes, Tabit ibn Korra, Hudde, and others.

As to the transliteration of Oriental names the author is no more at fault than most other historians. It is unfortunate that we have no board that can consider the whole matter with some semblance of authority and give us a standard system of diacritical marks and of the English equivalents of proper names as they appear in languages not using the Latin alphabet. If such forms as Ŝ, ṭ, ḥ, ī, and al- are to be used, as they are in this book, it is due to the reader

to have a key to assist him in their pronunciation. This is especially the case where, as here, they are not used uniformly, as witness Madjritī (p. 104), Majrītī (p. 104), and Majriti (p. 109).

The final point which it seems worth while to mention is concerned with proofreading and printer's errors. It appears ungenerous to refer to such little matters as lack of uniformity, misspelled words, carelessness in the use of accents, and the like; and yet, like insect stings, in the aggregate they are, to say the least, causes for regret. For example, it is perfectly proper to use either "geometric" or "geometrical," but it is annoying to see no uniformity in the matter, even on a single page,—and similarly for "analytic" and "analytical." Is the *Elements* of Euclid to be considered as a single treatise hence taking a singular verb, or should we say "the *Elements* are . . ."? On page 30 the word is considered as plural and on page 32 as singular. Is Claudius Ptolemy (p. 46) to be preferred to Claudius Ptolemaeus (p. 45), or should we have merely Ptolemy (p. 47) or, as in other cases, C. Ptolemy?

Of printer's errors there is a much larger number than usual, but it is hardly the proper business of the reviewer to single them all out. They are simply annoyances, hardly attributable to the author. It is merely pressroom carelessness that leads to a form like $mc \gtrless nd$, with the equality sign omitted (p. 32), to the use of the angle symbol for inequality (p. 143), to spelling *nor* for *not* (p. 122), and to the use of Tivolis' for Tivoli's (p. 123), and Trigonometry for Trigonometrie (p. 137 n.). That the name of Apollonius is misspelled on page 36 is not the author's fault, primarily; it is the blunder of the proofreader. The spelling of paradoxies for paradoxes (twice on p. 23) is also an error of the press. It would be a pleasure to say the same for the name of Maupertuis, which is mispelled every time it appears. This is akin to placing Bagdad on the Euphrates (p. 99) instead of on the Tigris. The printer's attempts to make up the three peculiar Greek numerals (pp. 52, 53) are not altogether happy, but the error in his use of the zeta for 6 (p. 52) is quite unpardonable, not to speak of his use of the final sigma for the same purpose (p. 53). Śrīdhara is made to appear as S'rīdhara and Śiromani as S'iromani (p. 85), although a crude effort is later made to conform to the proper style by using the form S'rīdhara (p. 94).

The student will probably not be troubled by the uncertainty in the title of Hankel's *Geschichte* (pp. 10 n., 57 n.), but it would have been better had it been given correctly in each case.

The name Vlacq appears as Vlack in two cases, which is allowable except for uniformity; Āryabhaṭa appears also (p. 85) as Āryabhata; "the Roman bishop Hippolytos" might perhaps better have the Roman termination to his name (p. 91); the degree sign is omitted in sin $(90 - \alpha)$ although given on the same page in sin $(90° - 2\gamma)$; the name Fine (Oronce) is correctly spelled on page 143 but appears incorrectly as Finé on page 116 (an error to which many of us must probably plead guilty); Coimbra appears as Coïmpre (p. 142 n.); Núñez or Nonius is called by the much less familiar name of Nunes (p. 142); Christopher Clavius appears as Christophorus Clavius (p. 144), a form which is rather un-

usual in English works, and also in the curious guise of C. S. Clavio (p. 184.) Mention should also be made of the genitive forms of certain proper names. Shall we expect Wallis' or Wallis's, where the name is not a plural form? There is authoritative sanction for either, but there is no sanction for using the two forms indiscriminately as is the case with various names throughout this work.

With respect to the index there should be mentioned the fact that it is unusually complete and that it has one decided improvement over most efforts of this kind. This improvement consists in giving the most important reference first, and in the use of subtitles under the heading entries, each of which is a great convenience. There are various omissions, such as Coss, Harun al-Rashid, Fine (O., but see Orontius), and Salviati, and there are a few misspelled names, like Durege for Durège, but on the whole there is no reasonable cause for complaint.

In closing this review it should be said once more that the purpose of the writer has not been to find fault with respect to small details. He has endeavored to call attention to the distinctively good features of the work and to state only the leading features that are likely to be subjects of adverse criticism. If the proper limits of a review of this kind had permitted, he would have been glad to discuss such questions as the Roman use of 120 as a diameter, the rise of the sexagesimal fraction, and the causes of the prominence of mathematics in various epochs and in various countries, and to commend specifically certain other features of value. He wishes, however, to add the statement that the book shows a wide reading of secondary material and a considerable study of original sources. It will, of course, have a place in every mathematical library of any importance and will prove a helpful work of reference, particularly with respect to the mathematics of the last two centuries.

DAVID EUGENE SMITH.

Report of the President of the Carnegie Institution of Washington for the year ending October 31, 1919. Washington, D. C., 1919. 4to. 97 pp.

For the mathematician perhaps the most interesting items in this report by President R. S. WOODWARD, a charter member of the Association, are the paragraph concerning the Hooker telescope on Mount Wilson (pages 16–17) which "under repeated tests has proved efficient quite beyond the conservative theoretical predictions of attainable capacities," and the following paragraph (pages 18–19):

"The most elementary, the most essential, and hence the most widely used, if not esteemed, of the sciences is arithmetic. It is a fundamental requisite, in fact, of all exact knowledge. Ability to add, subtract, multiply, and divide affords probably the simplest test of capacity for correct thinking. Conversely, inability or indisposition to make use of these simple operations affords one of the surest tests of mental deficiency, as witnessed, for example, by numerous correspondents who are unable to or who refuse to apply these operations to the finances of the Institution. But the familiar science of arithmetic lies at the foundation also of a much larger and a far more complex structure called the theory of numbers. This theory has been cultivated by many of the most acute thinkers of ancient and modern times. It has more points of contact with quantitative knowledge in general than any other theory except the theory of the differential and

integral calculus. These two theories are complementary, the first dealing with discrete or discontinuous numbers and the second with fluent or continuous numbers. Naturally, a subject which has attracted the attention of nearly all of the great mathematicians of the past twenty centuries has accumulated a considerable history. The more elementary contributions of Euclid, Diophantus, and others of the Greek school; the extensions of Fermat, Pascal, Euler, Newton, Bernoulli and many others in the seventeenth and the eighteenth centuries; and the work of Lagrange, Laplace, Gauss, and their numerous contemporaries and successors of the nineteenth century, make up an aggregate which has stood hitherto in need of clear chronological tabulation and exposition. This laborious task was undertaken about ten years ago by a Research Associate of the Institution, Professor Leonard E. Dickson, of the University of Chicago. A publication under the title 'History of the Theory of Numbers' has resulted, and volume I (8vo, xii + 486 pp.), devoted to divisibility and to primality of numbers, has appeared during the past year; and a second volume devoted to diophantine analysis is now in press. This work is remarkable for its condensation of statement. It contains more information per unit area than any other work issued thus far by the Institution. It is remarkable also for the care taken by the author and by his collaboration to secure precision and correctness, a number of experts having assisted in the arduous labors of verification required during the process of printing."

Two sections, completing volume 9, of the great *New English Dictionary on Historical Principles* (cf. *1919*, 256–257) were published by the Clarendon Press, Oxford, in August, 1919. They cover the portions Stratus-Styx, Sweep-Szmikite. The longest article in the sections is that on the verb 'strike' (29 columns). One interesting point here brought out is that the use of 'strike' in the sense 'to refuse to work' is an eighteenth century development from the nautical use in 'to strike a mast' (*Annual Register*, 1768; p. 92: "A body of sailors . . . proceeded . . . to Sunderland . . . and at the cross there read a paper, setting forth their grievances. . . . After this they went on board the several ships in that harbor and struck (lowered down) their yards, in order to prevent them from proceeding to sea.").

Other terms of interest to the mathematician (with indications of some of the earliest dictionary references) are: stream-line (in hydrodynamics, Maxwell; *Electricity and Magnetism*, 1873); strophoid (W. W. Johnson in *Amer. Journal of Math.*, 1880[1]); striction (line of striction, Frost, *Solid Geometry*, 1875, p. 297); sturmian (Sylvester, *Philosophical Trans.*, volume 143, 1853); style (Leybourn, *Cursus Mathematicus*, 1690, p. 704—the line shadow on the plane of the dial showing the true hour-line); symbolic, symbolical, symbolically; symmedian[2] (Casey, *Analytical Geometry*, 1885, pp. 45, 247); symmetral (Jeake, *Arithmetic*, 1674, p. 295 "Commensurable, called also Symmetral, is when the given Numbers have a Common Divisor" also "Symmetral Surdes" [obsolete uses]—Gurney, *Crystallography*, 1878, p. 27 "The two halves on either side of this symmetral plane are in all respects similar"); symmetric, symmetrical; symmetroid (Cayley's name for a certain surface of the fourth order, 1870, *Coll. Math. Papers*, Vol. 7, p. 134); symmetry; synchronism, synchronous curve (Brande and Cox, *Dict. Sci.*, 1867); synclastic (Thomson and Tait, *Nat. Phil.*, vol. 1, part 1, 1867, § 128 "We may divide curved surfaces into Anticlastic and Synclastic. A saddle

[1] The term strophoïde seems to have been used for the first time by Montucci in 1846 (*Nouv. Annales de Math.*, vol. 5, p. 470); it was called the logocyclic curve by James Booth (*Treatise on some new Geometrical Methods*, vol. 1, 1873, p. 295).

[2] The French equivalent of this term, "symédiane," was invented by M. D'Ocagne (*Nouv. Annales de Math.*, October, 1883, p. 451) as an abbreviation for "symétrique de la médiane."

gives a good example of the former class; a ball of the latter"); synectic (B. Williamson in *Encycl. Brit.* vol. 24, 1888, p. 72 "A function of a complex variable which is continuous one-valued, and has a derived function when the variable moves in a certain region of the plane is called by Cauchy Synectic in this region"); syntax; syntheme (Sylvester, 1844, *Coll. Math. Papers*, 1904, vol. 1, p. 91 "Let us agree to denote by the word syntheme any aggregate of combinations in which all the monads of a given system appear once and once only. . . . Let us begin with considering the case of duad synthemes"); synthetic; syntractrix and syntractory (G. Peacock, *Examples Diff. Calc.*, 1820 and G. Salmon, *Higher Plane Curves*, 1852); systatical (Jeake, *Arithmetic*, 1674, p. 662 "Three . . . is called a Systatical or Substantial Number, because all Sublimary Bodies consist of the three principal Substances, Sal, Sulphur, and Mercury") [obsolete]; syzygant; syzygetic and syzygy (Sylvester, *Cambr. and Dublin Math. Journal*, vol. 5, 1850, p. 276 "The members of any group of functions, more than two in number, whose nullity is implied in the relation of double contact, . . . must be in syzygy. Thus, *PQ*, *PQR*, *QR*, must form a syzygy").

It may be remarked that the editors of 'N.E.D.' have overlooked three words used in mathematical literature, namely: symptose, syntypic, and syrrhizoristic. The first two occur in the index to Cayley's *Coll. Math. Papers*, 1898, on p. 135. (The use of the term syntype, in natural history, *is* presented in 'N.E.D.'). Sylvester introduced the word syrrhizoristic (*Philosophical Trans.*, vol. 143, 1853, and *Coll. Math. Papers*, vol. 1, p. 585 "A syrrhizoristic series is a series of disconnected functions which serve to determine the effective intercalations of the real roots of two functions lying between any assigned limits").

While 'N.E.D.' lists several meanings of the word systatic, there is no reference to its use in mathematics. Have this word and asystatic (not in 'N.E.D.') ever been used as mathematical terms in English writings? They are familiar to the Frenchman (*Encyclopédie des sciences mathematique*, tome 2, volume 4, p. 224: "groupes systatiques et asystatiques"), to the German (Lie-Engel, *Transformationsgruppen*, Band 1, Leipzig, 1888, Kapitel 24: "Systatische und asystatische Transformationsgruppen," pp. 497–522), and to the Italian (L. Bianchi, *Lezioni sulla teoria dei gruppi continui*, Pisa, 1918, p. 185: "gruppi sistatici ed asistatici").

The Physical Society of London. *Report on the Theory of Gravitation.* By A. S. EDDINGTON, London, Fleetway Press, 1918. 8vo. 7 + 91 pp. Price, in paper, 6s. 3d.

Quotation from the Preface: "The relativity theory of gravitation in its complete form was published by Einstein in November, 1915. Whether the theory ultimately proves to be correct or not, it claims attention as one of the most beautiful examples of the power of general mathematical reasoning. The nearest parallel to it is found in the applications of the second law of thermo-dynamics, in which remarkable conclusions are deduced from a single principle without any inquiry into the mechanism of the phenomena; similarly, if the principle of equivalence is accepted, it is possible to stride over the difficulties due to ignorance of the nature of gravitation and arrive directly at physical results. Einstein's theory has been successful in explaining the celebrated astronomical discordance of the motion of the perihelion of Mercury, without introducing any arbitrary constant; there is no trace of forced argument about this prediction. It further leads to interesting conclusions with regard to the deflection of light by a gravitational field, and the displacement of spectral lines on the sun, which may be tested by experiment.

"The arrangement of this report is guided by the object of reaching the theory of these crucial phenomena as directly as possible. To make the treatment rather more elementary, use of the principle of least action and Hamiltonian methods has been avoided; and the brief account of these in Chapter VII is merely added for completeness. Similarly, the equations of electrodynamics are not used in the main part of the Report. Owing to the historical tradition, there is an undue tendency to connect the principle of relativity with the electrical theory of light and matter, and it seems well to emphasize its independence. The main difficulty of this subject is that it requires a special mathematical calculus, which, though not difficult to understand, needs time and practice to use with facility. In the older theory of relativity the somewhat forbidding vector products and vector operators constantly appear. Happily this can now be avoided altogether; but in its place we use the absolute differential calculus of Ricci and Levi-Civita."

Contents—I (Pages 1–13): The restricted principle of relativity; II (14–29): The relations of space, time, and force; III (30–40): The theories of tensors; IV (41–47): Einstein's law of gravitation; V (48–58): The crucial phenomena; VI (59–70): The gravitation of a continuous distribution of matter; VII (71–81): The principle of least action; VIII (82–91): The curvature of space and time.

NOTES.

The works of EVANGELISTA TORRICELLI, "edite in occasione del III centenario della nascita col concorso del Comune di Faenza da Gino Loria e Giuseppe Vassura," have been published in three volumes (about 1800 pages, Faenza, G. Montanari, 1919; price 60 lire). There is a valuable "Introduzione" (pages iii–xxxviii of the first volume) by Loria.

Vuibert (Paris) published in 1919 the first volume of a three-volume work by H. BROCARD and T. LEMOYNE entitled: *Courbes géométriques rémarquables (courbes spéciales) planes et gauches.* The volume contains 460 royal-octavo pages and is listed at 18 francs. It will soon be reviewed in this MONTHLY.

The Hydrographic Office, Washington, has recently published: *General Catalogue of Mariner's Charts and Books*, corrected to April 1, 1919 (293 pages). In Special Publication no. 60 of the U. S. Coast and Geodetic Survey, Mr. O. S. ADAMS makes *A Study of Map Projections in General* (24 pages). The author states that

"an attempt has been made to treat in simple form some of the fundamental ideas that underlie the subject of map projections in general. There has been no intention to develop any phase of the subject at any length, but merely to give briefly some suggestions under the different headings that, it is hoped, may be found helpful to those who wish to get an understanding of the subject."

The issue of *Nature* for November 6, 1919, was a "jubilee number" (84 pages), and contained about forty brief articles concerning progress in various phases of science. Sir Norman Lockyer, the founder of the journal, in November, 1869, wrote "Valedictory Memories," and H. Deslandres, director of the Astrophysical Observatory of Mendon, wrote the sketch of Sir Norman (of whom there is a fine portrait supplement) for the "Scientific Worthies" series. The article on "Science and the Church" is by Canon J. M. Wilson who states that he "was a fair mathematician" fifty years ago (he was a senior wrangler). Readers of *Euclid and his Modern Rivals* (1879, second edition, 1885) will recall that C. L. Dodgson (Lewis Carroll) and De Morgan found much to criticize in the Canon's

Elementary Geometry . . . following the Syllabus of Geometry prepared by the Geometrical Association.[1] His *Solid Geometry and Conic Sections* was familiar to a good many American students of mathematics of twenty-five years ago.

The *Journal für die reine und angewandte Mathematik* was founded by Crelle in 1826 with Gergonne's *Annales de mathématiques pures et appliquées* as a model. Success in the undertaking was partly assured through an arrangement he made with the Kultusministerium whereby he was allowed to add to the title of the *Journal* the words to be found even in volume 146, 1915: "Mit tätiger Beförderung hoher Königlich Preussischer Behörden." (It is a sign of the times that this legend does not appear in volume 149, 1919.) The "Beförderung" consisted on the one hand in issuing strong official recommendation of the *Journal* not only to universities, and institutions of a similar nature, but also, for example, to government boards and, through Prussian ambassadors, to foreign countries. On the other hand the "Beförderung" involved the purchase of a number of copies of the *Journal* which were distributed to various schools—a custom prevailing to very recent times. In this way Weierstrass, for example, while a gymnasium pupil, received inspiration by discovering an uncut copy of the *Journal* "mit den schönen Abhandlungen von Steiner, von denen auch ein Primaner etwas verstehen konnte."

ARTICLES IN CURRENT PERIODICALS.

ALUMNI BULLETIN, College of St. Thomas, St. Paul, Minn., volume 3, no. 1, February, 1919: "Rev. William Earnshaw Etzel," 14–15 [full page portrait, 14; died February 3, 1919; charter member of the Mathematical Association of America].

AMERICAN CATHOLIC QUARTERLY REVIEW, Philadelphia, volume 44, January, 1919: "Catholic church and the gentle science of numbers" by E. Von R. Wilson, 121–145.

AMERICAN MACHINIST, New York, volume 51, October 9, 1919: "The equation of the involute simplified" by N. Finkelstein, 693–694; "Cam design and construction" by F. De R. Furman, 695–698; "A little question in trigonometry" by K. H. Condit, 713–714.

AMERICAN STATISTICAL ASSOCIATION, Quarterly Publications, Boston, volume 16, September, 1919: "On functional relations for which the coefficient of correlation is zero" by H. L. Rietz, 472–476.

BULLETIN DES SCIENCES MATHÉMATIQUES, volume 54, May, 1919: Review by E. Picard of G. H. Halphen's *Oeuvres*, tome 2 (Paris, 1918), 105; "Mémoire relatif à l'étude des substitutions rationnelles à une variable" by G. Julia, 106–109; Review by G. Bigourdan of E. Belot's *L'origine des formes de la terre et des planètes* (Paris, 1918), 110–112; Review by R. le Vavasseur of F. X. Kugler's *Die Babylonische Mondrechnung* (Freiburg 1. B., 1900), *Sternkunde und Sterndienst in Babel-Assyriologische, astronomische und astralmytologische Untersuchungen* (Münster i. W., 1907–1913), and *Im Bannkreis Babels* (Münster i. W., 1910), 112–118; Review by E. Cahen of O. Hölder's *Die Arithmetik in Strenger Begründung* (Leipzig, 1914), 118–121; Review by P. du Plessis of H. von Sanden's *Praktische Analysis* (Leipzig, 1914), 121–124; "La série $\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{29} + \frac{1}{31} + \frac{1}{41} + \frac{1}{43} + \frac{1}{59} + \frac{1}{61} + \cdots$ où les dénominateurs sont 'nombres premiers jumeaux' est convergente ou finie" (suite et fin) by V. Brun, 124–128.—June: Review of E. J. Boudin's *Leçons de calcul des probabilites* faites à l'Université de Gand de 1846 à 1890, publiées avec des Notes et des additions par P. Mansevi (Paris, 1916), 129–133; Review by G. Loria of P. Duhem's *Système du monde*, tome 5 (Paris, 1917), 133–135; Review by E. Lebon of F. Frenet's *Recueil d'exercices sur le calcul infinitésimal* (seventh edition, Paris, 1917), 135–136; Review by E. Ouivet of A. Schoenflies's *Entwicklung der Mengenlehre und ihrer Anwendungen*, part 1 (Leipzig,

[1] That is, the A. I. G. T. (Association for the Improvement of Geometrical Teaching) afterwards the Mathematical Association.

1913), 136–143; Review by C. Bioche of W. Killing and H. Hovestadt's *Handbuch des Mathematischen Unterrichts*, Band 2 (Leipzig, 1913), 143–144.—July: Review by G. Giraud of Goursat's *Cours d'analyse mathématique*, tome 2 (third edition, Paris, 1918), 145–146; Review by R. le Vavasseur of C. E. Cullis's *Matrices and Determinoids*, volume 2 (Cambridge, 1918), 146–149; Review by P. Mansion of R. E. Moritz's *Memorabilia Mathematica* (New York, 1914), 150–159; Review by G. Cotty of Klein and Sommerfeld's *Ueber die Theorie des Kreisels*, Heft 1, second edition (Leipzig, 1914), 159–161.

Bulletin of the American Mathematical Society, volume 26, no. 2, November, 1919: "The twenty-sixth summer meeting of the American Mathematical Society" by E. J. Moulton, 49–66; "Form of the number of subgroups of prime power groups" by G. A. Miller, 66–72; "On the rectifiability of a twisted cubic" by T. Hayashi, 73–75; "Some generalizations of the satellite theory" by R. M. Winger, 75–79; Review by G. A. Miller of F. Cajori's *A History of Mathematics* (2d edition, New York, 1919), 79–85; Reviews by R. D. Carmichael of W. Ahrens's *Mathematische Spiele* (3d edition, Leipzig and Berlin, 1916) and of Por J. de Mendizabal Tamborrel's *Tratado Elemental de Goniometria* (2d edition, Mexico, 1917), 86; Notes, 87–93; New Publications, 94–96.

Contemporary Review, volume 116, December, 1919: "Einstein's theory of space and time" by A. S. Eddington, 639–643.

Educational Review, volume 58, October, 1919: "The new comedy of errors" by R. E. Moritz, 219–238 [Motto: "'Every absurdity has a champion to defend it, for Error is always talkative,' Goldsmith." First paragraph: "It is no uncommon experience for a teacher who stresses mental training and the formation of proper habits of thought to be reminded by such of his colleagues as have enjoyed the advantages of recent courses in pedagogy that he is educationally behind the times; that the doctrine of mental training is a superstition or an 'exploded myth'; that mental faculties can not be stimulated or strengthened, since modern psychology has shown that no such faculties exist; that modern educationists recognize the 'basis of content' as the only sound basis of education, and other similar catch phrases which permeate the educational literature of recent years."]—December, 1919: "The contribution of mathematics to world progress" by C. N. Moore, 413–419 [First paragraph: "One of the compensating features of a great war may be found in the light that it sheds on relative values. A nation under the stress of battle for its existence or its dearest principles seeks the most efficient way of doing things with a directness that is very rare in time of peace. Thus it came about during recent months that persons with varying degrees of mathematical knowledge were in great demand for war activities of the first moment. Some superficial observers have drawn from this fact the totally false conclusion that the applications of mathematics in the conduct of modern warfare are far more numerous and more important than its application to the tasks of peace. They have overlooked the real explanation, namely, that the tremendous urge of war necessity secured for mathematics a wide-spread recognition of its true importance in many technical and scientific undertakings."]

The Journal of Educational Psychology, volume 10, nos. 5–6, May–June, 1919: "The effects of special drill in arithmetic as measured by the Woody and the Courtis arithmetic tests" by J. E. Evans and Florence E. Knoche, 263–276.

The Monist, volume 29, no. 4, October 1919: "Indefinables and indemonstrables in mathematics and theology" by P. E. B. Jourdain, 547–559; "Lotze's theory of the subjectivity of time and space" by J. E. Turner, 579–600; "Professor Russell's infinite" by H. H. Williams, 616–619; "Infinity and the part-and-whole axiom" by H. M. Gordon, 619–629.

Nature, volume 104, no. 2608, October 23, 1919: Review by G. B. M[athews] of H. B. Hedrick's *Interpolation Tables* (Washington, 1918), 152.

La Nature, volume 46, September 20, 1919: "Comment déchiffrer les textes en langage secret. La cryptophotie" by N. Flamel, 188–192—September 27: "Une machine a tracer les courbes" by H. Volta [Quotation: "Dans un article consacré au Bathyrhéomètre de M. J. Delage nous avions indiqué le principe du curieux appareil appelé par Lord Kelvin le '*tide predicter*' et qui permet de réaliser mécaniquement les compositions d'un numbre quelconque de fonctions harmoniques. Mais cet instrument dont la principale application est la détermination de la marée d'un lieu, est susceptible d'autres emplois, car, légèrement modifié, il trace mathématiquement avec une grande régularité des courbes de natures très diverses. C'est la machine à décrire des courbes due à M. Rigge que nons allons étudier d'après *Scientific American*." The article in question was "Compound harmonic motion" by W. F. Rigge, *Scientific American Supplement*, February 9–16, 1918, vol. 85, pp. 88–91, 108–110.]

NEW MEXICO NORMAL UNIVERSITY BULLETIN, no. 62, cover date November, Bulletin date April, 1919: *An experiment in the teaching of mathematics to high school students* by T. G. Rodgers, 8 pages + cover. [Portrait of author, cover 2. Note by President F. H. H. Roberts, page 1: "Believing that there could be an improvement in the method of presenting mathematics to high school pupils, Professor Rodgers undertook an experiment that is outlined in these pages. The results have been more than satisfactory to the administration. Correspondence and criticisms are solicited." Page 2: "President Roberts, in compliance with your request for information in regard to the experimental work in the teaching of secondary mathematics which the Department of Mathematics has been carrying on for the past five years, I submit the following report:

"An effort to add a mite to the solution of these four problems has been attempted: 1. to vitalize the teaching of secondary mathematics; 2. to reduce the large number of failures in mathematics; 3. to vitalize mathematical reviews; and 4. to make the examination test and reward ability without being unjust to the faithful plodder."]

NINETEENTH CENTURY AND AFTER, volume 86, December, 1919: "The new theory of gravity" by O. J. Lodge, 1189–1201.

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 78, July, 1919: "Sur quelques courbes associées à une classe d'hélices cylindriques" by M. Egan, 241–248; "Concours d'agrégation de 1913. Solution de la question de mécanique" by M. de Sparre, 248–268; "Certificats de Licence (Marseille, mars 1919,)," 269–276; "Questions and solutions," 277–280.

PROCEEDINGS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES, volume 55, no. 2, December, 1919: "The functional relation of one variable to each of a number of correlated variables determined by a method of successive approximation to group averages: a contribution to statistical methods" by G. F. McEwen and E. L. Michael with an introduction: "A step forward in the methodology of natural science" by W. E. Ritter, 91–133, + 2 tables. [The paper was presented in abstract before the San Francisco section of the American Mathematical Society in 1917.]

REVUE DES DEUX MONDES, November 1, 1919: "Au royaume de l'infiniment petit" by C. Nordmann, 214–225.

REVUE GÉNÉRALE DES SCIENCES PURES ET APPLIQUÉES, volume 30, October 15, 1919: "La mesure absolue de l'intensité du son" by A. G. Webster, 547–550; Review by P. Appell of G. Bouligand's *Cours de géométrie analytique* (Paris, 1919).

SCIENCE PROGRESS, volume 14, October, 1919: Recent advances in science—mathematics, 189–196; Reviews by P. E. B. Jourdain of J. B. Shaw's *Synopsis of Linear Associative Algebra* (Washington, 1907), T. R. Running's *Empirical Formulas* (New York, 1917), C. Davison's *Differential Calculus for Schools and Colleges* (London, 1919), and L. Silberstein's *Projective Vector Algebra: an Algebra of Vectors independent of the Axioms of Congruence and of Parallels* (London, 1919), 340–343; Review by H. S. Jones of H. Jacoby's *Navigation*, second edition, (New York, Macmillan, 1918) and of G. L. Hosmer's *Textbook on Practical Astronomy* (New York, 1917), 343–344; Review by J. Rice of R. C. Tolman's *Theory of the relativity of motion* (Berkeley, 1918), 346–347.

AMERICAN DOCTORAL DISSERTATIONS.

S. W. Reaves, *Metric properties of flecnodes on ruled surfaces*, [Reprinted from *Giornale di Matematiche di Battaglini*, volume 55, 1917], 28 pp. + 1 page of errata; [the article was published without the author's correction having been inserted]. (Chicago, 1915.)

L. L. Thurstone, *The learning curve equation*, [Reprinted from *Psychological Monographs*, vol. 26, Princeton, N. J.], 1919. 4 + 51 pp. (Department of Psychology, Chicago University, 1918.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

[Send all communications about problems and solutions to B. F. FINKEL, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

2814. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

The bisectors of the angles formed by the diagonals of an inscribed quadrilateral are: (1) parallel to the lines joining the midpoints of the arcs subtended by the opposite sides of the quadrilateral on its circumcircle; (2) parallel to the bisectors of the angles formed by any pair of opposite sides of the quadrilateral; (3) equally inclined to pairs of sides of the quadrilateral.

2815. Proposed by the late L. G. WELD.

A right circular cone is laid upon an inclined plane so that its element of contact makes a given angle with the slant line of the plane. Assuming that there is no slipping and that the rolling friction is negligible, find the time of oscillation of the cone.

2816. Proposed by W. H. ECHOLS, University of Virginia.

Find two points D and E on the sides AB and CB, respectively, of a triangle ABC such that $AD = DE = EC$.

Give a rule and compass construction.

2817. Proposed by W. D. CAIRNS, Oberlin College.

The normal probability curve is sometimes called the binomial curve from the correspondence of its ordinates to the terms of $(1 + 1)^k$. Find an expression for the "mean deviation" of these terms from the median for the case where $k = 2m$. Assume the scale division as unity.

2818. Proposed by S. A. COREY, Des Moines, Iowa.

What is the maximum error that could occur in computing the common logarithm of $(1 + x)$ by the formula:

$$\text{Log}_{10}(1 + x) = .144,764,827\,\frac{x}{1+x} + .579,059,309\,\frac{x}{2+x} + .072,151,17\,\frac{x^2}{1+x},$$

where $0 \leqq x \leqq 6/10$?

2819. Proposed by B. F. FINKEL, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle with a given base and a given vertical angle.

2820. Proposed by C. B. HALDEMAN, Ross, Ohio.

Given one angle and the radii of the inscribed and circumscribed circles, to construct the triangle geometrically.

2821. Proposed by FRANK IRWIN, University of California.

The quantities $x_1, x_2, \cdots x_n$ vary in such a way that their sum (or any other one of the elementary symmetric functions) remains constant; investigate the maxima and minima of the remaining elementary symmetric functions.

SOLUTIONS OF PROBLEMS.

272 (Mechanics) [1913, 64; 1919, 213]. Proposed by J. F. LAWRENCE, Stillwater, Okla.

A perfectly rough circular cylinder is fixed with its axis horizontal. A sphere is placed on it in a position of unstable equilibrium, and projected with a given velocity parallel to the axis of

the cylinder. If the sphere be slightly disturbed in a horizontal direction perpendicular to the direction of the axis of the cylinder, determine at what point the sphere will leave the cylinder.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

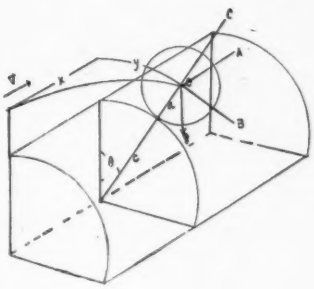
Assume as moving axes the following mutually perpendicular lines through the center of the sphere: GA parallel to the axis of the cylinder, GC along the common normal and GB. Let the velocity of the sphere parallel to GA be u and to GB be v. Then if ω_3 be the angular velocity of the sphere about GC and ρ the radius of curvature of the circular section of the surface upon which the center of the sphere G moves, we have the equations of motion[1]

$$\frac{du}{dt} = \frac{a^2}{a^2+k^2}X - \frac{k^2}{a^2+k^2}\frac{a\omega_3 v}{\rho}, \tag{1}$$

$$\frac{dv}{dt} = \frac{a^2}{a^2+k^2}Y, \tag{2}$$

and

$$\frac{ad\omega_3}{dt} = \frac{uv}{\rho}, \tag{3}$$



X and Y being the impressed forces acting on the sphere parallel to GA and GB, and a and k the radius and radius of gyration of the sphere, respectively. Assuming the sphere to be of unit mass and the cylinder of radius c, we have

$$X = 0 \qquad Y = g\sin\theta, \tag{4}$$

$$\rho = a + c, \tag{5}$$

$$y = (a+c)\theta, \quad \text{(see figure)}, \tag{6}$$

$$v = (a+c)\dot{\theta}, \tag{7}$$

$$k^2 = (2/5)a^2.$$

By these, we get from (1), (2), and (3)

$$du/dt = -(2/7)a\omega_3\dot{\theta}, \tag{8}$$

$$(c+a)\ddot{\theta} = (5/7)g\sin\theta, \tag{9}$$

$$ad\omega_3/dt = u\dot{\theta}. \tag{10}$$

By (8) and (10), $udu + (2/7)(a\omega_3)d(a\omega_3) = 0$. Hence

$$u^2 = -(2/7)(a\omega_3)^2 + V^2, \tag{11}$$

for $u = V$ when $\omega_3 = 0$, V being the initial velocity of projection. By (10) and (11), $ad\omega_3/dt = \sqrt{V^2 - (a\omega_3)^2}\,\tfrac{2}{7}d\theta/dt$; whence, $\sin^{-1}(\sqrt{2/7}a\omega_3/V) = \sqrt{2/7}\,\theta$, for $\omega_3 = 0$ when $\theta = 0$, or

$$\sqrt{2/7}a\omega_3 = V\sin\sqrt{2/7}\theta. \tag{12}$$

By (12) and (11), $u^2 = V^2 - V^2\sin^2\sqrt{2/7}\theta = V^2\cos^2\sqrt{2/7}\theta$. Hence,

$$u = V\cos\sqrt{2/7}\theta. \tag{13}$$

If R is the reaction between the sphere and the cylinder, we have $g\cos\theta - R = (c+a)\dot{\theta}^2$, and when the sphere leaves the cylinder $R = 0$, or

$$g\cos\theta = (c+a)\dot{\theta}^2. \tag{14}$$

By (9), $(c+a)(\dot{\theta}^2/2) = (5/7)g(1-\cos\theta)$; so that by (14) when they separate,

$$g\cos\theta = (10/7)g(1-\cos\theta),$$

or $\cos\theta = 10/17$. The distance traversed parallel to the axis of the cylinder from the position of slight displacement is

$$x = \int u\,dt = \int \frac{u}{\dot{\theta}}\,d\theta = \frac{V}{10}\sqrt{35}\sqrt{\frac{c+a}{g}}\int_{\theta_1}^{\cos^{-1}\frac{10}{17}} \cos\sqrt{2/7}\theta\,\csc\frac{\theta}{2}\,d\theta,$$

by (12) and (14), θ_1 being the initial angle of slight displacement.

[1] *Cf.* Routh, *Advanced Rigid Dynamics*, sixth edition, 1905, p. 172, article 225.—EDITORS.

277 (Mechanics) [1913, 196; 1919, 213]. Proposed by W. J. GREENSTREET, Burghfield Common, Berks., England.

Around a smooth fixed circular pulley is wound a massless inextensible string, and straight portions go to two free ends A and B to which masses are fastened. The mass at A is initially projected perpendicular to the string while the other is initially at rest. The length of the straight portion to the first mass is initially l and subsequently is r. Find the velocity of the second mass at that moment.

Solution by J. B. Reynolds, Lehigh University.

Let the masses initially at A and B be n and n', respectively. Let z be the distance n' has moved and r the distance from the pulley to n at any subsequent time; θ, the angle that r makes with its initial position l. If T is the tension in the string, the equations of motion are

$$T = -n(\ddot{r} - r\dot{\theta}^2), \qquad \frac{d}{dt}(r^2\dot{\theta}) = 0, \qquad T = n'\ddot{z}.$$

The second of these gives, $r^2\dot{\theta} = \text{const} = v_0 l$, v_0 being the initial velocity of projection. Again, since $r = z + \text{constant}$, $\ddot{z} = \ddot{r}$ and eliminating T between the first and third equations, we have

$$(n + n')\ddot{r} = nr\dot{\theta}^2 = \frac{nv_0^2 l^2}{r^3},$$

or, since

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr},$$

we get upon integrating,

$$(n + n')\dot{r}^2 = -\frac{n v_0^2 l^2}{r^2} + nv_0^2 = nv_0^2\left(1 - \frac{l^2}{r^2}\right).$$

Therefore, since the velocity $\dot{z}$ of $n' = \dot{r}$ at any time, we have

$$\dot{z} = \frac{v_0}{r}\sqrt{\frac{n}{n + n'}}\sqrt{r^2 - l^2}.$$

2700 [1918, 215; 1919, 215, 365]. Proposed by the late ARTEMAS MARTIN.

In a factory 250 men are paid an average wage of \$15 each per week. The men are paid unequally, the wages being \$20, \$16, \$10, and \$8 per week respectively, for different classes of work. How many are employed at each rate of pay?

Note.—I am told that this question was set in a Civil Service examination paper to be worked by arithmetic. 2,896 answers have been found. Are there any more?

III. Remarks by H. S. Uhler, Yale University.

On page 366 of the October issue the interesting suggestion is made by Professor D. N. Lehmer that: "The actual number might be obtained by counting the number of lattice points in this quadrilateral." I have followed this suggestion by counting along lines parallel to the axis of ordinates. Let the sides of the trapezium the equations of which are $3X + 2Y - 625 = 0$, $X - Y = 0$, $875 - 4X - 2Y = 0$, and $Y = 0$ be denoted respectively by I, II, III, and IV. The coördinates of the vertices are (125, 125), ($145\frac{5}{6}$, $145\frac{5}{6}$), ($208\frac{1}{3}$, 0), and ($218\frac{3}{4}$, 0). The number of lattice points limited by sides I and II is $1 + 3 + 6 + 8 + 11 + 13 + 16 + 18 + 21 + 23 + 26 + 28 + 31 + 33 + 36 + 38 + 41 + 43 + 46 + 48 + 51 = 541$. The number limited by sides I and III is $2(52 + 51 + \cdots + 23 + 22) + 21 = 2315$. The number limited by sides IV and III is $20 + 18 + \cdots + 4 + 2 = 110$. The total number of lattice points within and on the periphery of the trapezium is 2966, which is identically the value already given [*1919*, 216]. The numbers of points on sides I, II, III, and IV are 42, 21, 0, and 10, respectively. The sum of the distinct points, 72, verifies my earlier remark [*1919*, 216] that: "The number of solutions involving one or more zeros is 72." I should like to know whether there is any general significance to the fact that the "error" 36 [$2966 - 2929\frac{11}{16}$] is exactly one-half of 72, *i.e.*, the total number of peripheral solutions.

2726 [1918, 303]. Proposed by E. H. MOORE, University of Chicago.

Let $\alpha_1(x, y)$, $\alpha_2(x, y)$, $\alpha_3(x, y)$, $\alpha_4(x, y)$, $\kappa_1(x, y)$, $\kappa_2(x, y)$ be six real-valued continuous functions of (x, y) over the unit-square S: $(0 \leqq x \leqq 1;\ 0 \leqq y \leqq 1)$. Let α_1, α_2, α_3, α_4 be symmetric functions and of positive type, *i.e.*, for every real-valued continuous function $\xi(x)$ $(0 \leqq x \leqq 1)$

$$\int_0^1 \int_0^1 \xi(x)\alpha_\sigma(x, y)\xi(y)dxdy \geqq 0.$$

Prove the inequality:

$$(J_{15}'J_{26}''J_{37}'''J_{48}^{iv} - J_{16}'J_{25}''J_{35}'''J_{47}^{iv} - J_{25}'J_{46}''J_{17}'''J_{38}^{iv} + J_{26}'J_{48}''J_{15}'''J_{37}^{iv})\kappa_1\kappa_2\kappa_1\kappa_2 \geqq 0.$$

Here on the left there are four terms of which the first is

$$\int_0^1 \cdots \int_0^1 \alpha_1(u_1u_5)\alpha_2(u_2u_6)\alpha_3(u_3u_7)\alpha_4(u_4u_8)\kappa_1(u_1u_2)\kappa_2(u_3u_4)\kappa_1(u_5u_6)\kappa_2(u_7u_8)du_1 \cdots du_8;$$

thus the eight variables of integration $u_1 \cdots u_8$ are in order the eight arguments of the four functions $\kappa_1\kappa_2\kappa_1\kappa_2$, while the "integration" symbols J indicate how the variables are to be supplied as arguments to the four functions $\alpha_1\alpha_2\alpha_3\alpha_4$, *e.g.*, J_{26}'' indicates that α_2 (the superscript $''$ determining the subscript 2) has the arguments u_2 u_6.—Indicate another inequality of this type and determine the number of such inequalities.

SOLUTION BY C. F. GUMMER, Queen's University.

It has been shown by Hilbert (see Goursat, *Cours d'analyse mathématique*, tome 3, chap. XXXII) that for every real symmetric function $\alpha_\sigma(x, y)$ continuous in S there exist an orthogonal sequence of real functions $\phi_{\sigma 1}(x)$, $\phi_{\sigma 2}(x)$, $\cdots$, and a sequence of constants $c_{\sigma 1}, c_{\sigma 2}, \cdots$ (not zero) such that every function of the form $\int_0^1 \alpha_\sigma(x, y)\beta(y)dy$, where $\beta(y)$ is real and continuous in $(0 \leqq y \leqq 1)$, can be represented by the series

$$\sum_i \left[c_{\sigma i} \int_0^1 \phi_{\sigma i}(y)\beta(y)dy \cdot \phi_{\sigma i}(x) \right]$$

converging absolutely and uniformly in $(0 \leqq x \leqq 1)$. The function α_σ is of positive type if, and only if, $c_{\sigma i} > 0$ $(i = 1, 2, \cdots)$. The argument of Goursat, with little modification, shows that the convergence is uniform with respect to $(x, u, v, \cdots)$ in $(0 \leqq x \leqq 1,\ 0 \leqq u \leqq 1, \cdots)$ when for $\beta(y)$ is substituted $\gamma(y, u, v, \cdots)$ continuous in this region. This remark justifies the successive integration of the series term by term.

Applying this transformation to the first term of the inequality, we find in succession

$$\int_0^1 \alpha_1(u_1u_5)\kappa_1(u_5u_6)du_5 = \sum_i \left[c_{1i} \int_0^1 \phi_{1i}(u_5)\kappa_1(u_5u_6)du_5 \cdot \phi_{1i}(u_1) \right],$$

$$\int_0^1 \int_0^1 \alpha_1(u_1u_5)\kappa_1(u_5u_6)\alpha_2(u_2u_6)du_5du_6 = \sum_i \sum_j \left[c_{1i}c_{2j} \int_0^1 \int_0^1 \phi_{1i}(u_5)\kappa_1(u_5u_6)\phi_{2j}(u_6)du_5du_6 \cdot \phi_{1i}(u_1)\phi_{2j}(u_2) \right].$$

Since the cycle $u_1u_5u_6u_2$ is to be completed, we multiply by $\kappa_1(u_1u_2)$ and integrate with respect to u_1 and u_2, obtaining

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \alpha_1(u_1u_5)\alpha_2(u_2u_6)\kappa_1(u_1u_2)\kappa_1(u_5u_6)du_1du_2du_5du_6 = \sum_i \sum_j [c_{1i}c_{2j}\{K_1(\phi_{1i}\phi_{2j})\}^2],$$

where $K_i(\phi, \psi)$ stands for $\int_0^1 \int_0^1 \phi(u)\kappa_i(uv)\psi(v)dudv$. A similar treatment of the other four factors of the integrand shows that the first term is

$$\sum_i \sum_j \sum_k \sum_l c_{1i}c_{2j}c_{3k}c_{4l}\{K_1(\phi_{1i}\phi_{2j})K_2(\phi_{3k}\phi_{4l})\}^2.$$

The other terms of the inequality may be treated in a similar way, except that the second and third involve longer cycles, that of the second being $u_1u_6u_5u_3u_4u_7u_8u_2$. The left member is then found to be

$$\sum_i \sum_j \sum_k \sum_l c_{1i}c_{2j}c_{3k}c_{4l}\{K_1(\phi_{1i}\phi_{2j})K_2(\phi_{3k}\phi_{4l}) - K_1(\phi_{3k}\phi_{1i})K_2(\phi_{4l}\phi_{2j})\}^2,$$

which is clearly $\geqq 0$.

The number of such inequalities may be found from the coefficient of $c_{1i}c_{2j}c_{3k}c_{4l}$ in the iterated series. The first subscripts of the ϕ's in the first term of the squared factor admit 24 permutations, while the second term may be formed from it in four ways, each of the operators K_1K_2 retaining one of the ϕ's associated with it in the first term and in the opposite position. The 96 results so indicated reduce however to 48 by reason of the permutability of the terms of the squared factor. There are then 48 inequalities of precisely the same type as the one in question; but there are others of the same general type corresponding to any two permutations of the subscripts 1, 2, 3, 4, in the terms of the squared factor, making in all $\binom{24}{2}$ or 276 results, exclusive of the useless one $0 \geqq 0$ which is obtained when the same permutation is used twice. Of these, 24 are reducible inequalities of the type

$$(J_{15}'J_{26}''J_{37}'''J_{48}^{iv} - J_{15}'J_{26}''J_{38}'''J_{47}^{iv} - J_{15}'J_{26}''J_{47}'''J_{38}^{iv} + J_{15}'J_{26}''J_{48}'''J_{37}^{iv})_{\kappa_1\kappa_2\kappa_1\kappa_2}$$
$$= (J_{15}'J_{26}''{}_{\kappa_1\kappa_1}) \times (J_{37}'''J_{48}^{iv} - J_{38}'''J_{47}^{iv} - J_{47}'''J_{38}^{iv} + J_{48}'''J_{37}^{iv})_{\kappa_2\kappa_2} \geqq 0,$$

in which also each factor of the left member is separately $\geqq 0$. The other 204 are irreducible inequalities not isomorphic with the proposed formula, and falling into five classes which may be indicated by the permutation of first subscripts (*abcd*) which would be present in the second term of

$$K_1(\phi_1\phi_2)K_2(\phi_3\phi_4) - K_1(\phi_a\phi_b)K_2(\phi_c\phi_d)$$

(second subscripts being omitted) for a typical case of each class. That is, 96 correspond to the permutation (1342), 48 to (1432), 24 to (3412), 24 to (3421), and 12 to (2143).

2761 [1919, 124]. Proposed by W. W. DENTON, University of Michigan.

Find the lengths of the side of an equilateral triangle whose vertices are at given distances a, b, c from a given point.

I. Solution by C. E. Mange, Junior, Washington University.

To construct the triangle.

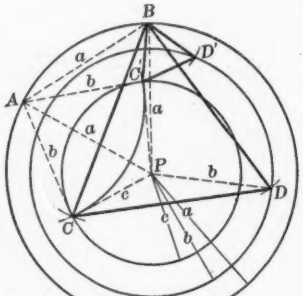
With given point, P, as a center, describe circles with radii a, b, and c, respectively. Assume $a > b > c$. Construct chord AB in circle (a) equal to a, and with A as a center describe an arc with radius b intersecting circle (c) in C and C'. With B as a center describe an arc of radius BC intersecting circle (b) in D.[1] Draw BC, CD, and DB. The triangle BCD is equilateral and fulfils the required conditions. (Similarly for the triangle $BC'D'$.)

Proof: Draw AC, AP, AB, BP, CP, and DP.

Now $AB = BP = PA$, by construction; therefore, $\triangle ABP$ is equilateral, and $\angle PBA = 60°$.

Since $\triangle ACB = \triangle BPD$, $\angle ABC = \angle PBD$; and hence $\angle ABC + \angle CBP = 60° = \angle CBD$. And since $CB = BD$, by construction, CBD is equilateral.

Area of



$$ACP = A = \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)},$$

$$\angle BAP = 60°, \quad \text{and} \quad \angle PAC = \arcsin\frac{2A}{ba}.$$

In triangle BAC,

$$BC = \sqrt{b^2 + a^2 - 2ba\cos\left[60° + \arcsin\frac{2A}{ba}\right]}$$

$$= \sqrt{b^2 + a^2 - 2ba\left[\frac{1}{2}\cdot\frac{b^2+a^2-c^2}{2ba} - \frac{\sqrt{3}}{2}\cdot\frac{2A}{ba}\right]}$$

$$= \sqrt{\frac{a^2+b^2+c^2}{2} + 2\sqrt{3}A}.$$

$$BC' = \sqrt{\frac{a^2+b^2+c^2}{2} - 2\sqrt{3}A}.$$

[1] D is that intersection which lies on the opposite side of BP from A, so that the four lines from B are in the order BA, BC, BP and BD. D' is the intersection determined in the same way when an arc of radius BC' is described.—Editors.

II. Solution by the Proposer.

Consider first the case in which the plane of the desired triangle passes through the given point, and the problem is confined to two dimensions. Let P be the given point, $a \geq b \geq c \geq 0$ the given distances, and e the length of the side of the required triangle. Let the angles between the lines drawn to the vertices be A, B, C, so that $A + B + C = 2\pi$ [2] and, therefore, one has

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2 \cos A \cos B \cos C.$$

From this and the law of cosines, $e^2 = a^2 + b^2 - 2ab \cos C$, etc., comes the equation,

$$e^4 - (a^2 + b^2 + c^2)e^2 + a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 = 0,$$

and the values e^2,

$$e^2 = \tfrac{1}{2}(a^2 + b^2 + c^2 \pm \sqrt{\Delta_0}),$$

where

$$\Delta_0 = 3(a + b + c)(a + b - c)(c + a - b)(b + c - a).$$

The values of e^2 are real and unequal, if, and only if, the relation $b + c > a$ is satisfied; they are real and equal if, and only if, the relation $b + c = a$ is satisfied. Moreover, the values of e^2 are never negative; this may be shown by using the relation

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2.$$

Therefore, there are two, one, or no equilateral triangles having their vertices at the given distances a, b, c, from a given point lying in their plane, according as these distances themselves may be taken as the lengths of the sides of a non-degenerate triangle, a degenerate triangle, or no triangle, respectively.

It can be shown that the given point always lies outside or at the vertex of one found triangle, and that when a^2 is between $b^2 + bc + c^2$ and $(b + c)^2$, it lies outside both triangles. (It is, therefore, not true, as stated in G. R. Perkins's *Plane and Solid Geometry*, New York, 1860, p. 231, that the minus sign must be used before the radical in the above formula when the given point lies outside the triangle.)

In the general case, some variable parameter beside the given lengths must be introduced: let this be the distance p from the given point P to the plane of the desired triangle. The projections of a, b, c, on this plane are

$$(1) \qquad a' = \sqrt{a^2 - p^2}, \qquad b' = \sqrt{b^2 - p^2}, \qquad c' = \sqrt{c^2 - p^2}.$$

The length of the side of an equilateral triangle lying in this plane and having vertices at the distances a', b', c', from the foot of this perpendicular is the length sought, and may therefore be obtained by putting a', b', c', in place of a, b, c, viz.,

$$e^2 = \tfrac{1}{2}(a^2 + b^2 + c^2 - 3p^2 \pm \sqrt{\Delta_p}),$$

where

$$\Delta_p = 9p^4 - 6p^2(a^2 + b^2 + c^2) + \Delta_0 = (3p^2 - \alpha)(3p^2 - \beta),$$

$$\alpha = a^2 + b^2 + c^2 - \sqrt{2}\sqrt{(a^2 + b^2 + c^2)^2 - \Delta_0} = a^2 + b^2 + c^2 - \sqrt{2}\sqrt{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2},$$

and

$$\beta = a^2 + b^2 + c^2 + \sqrt{2}\sqrt{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2}.$$

These values of e^2 are real (for real values of p^2), if, and only if, p^2 lies in one of the intervals, $p^2 \leq \tfrac{1}{3}\alpha$, $\tfrac{1}{3}\beta \leq p^2$.

For the first interval, e^2 is never negative, but p^2 is negative except when the condition $b + c \geq a$ is satisfied. For the second interval, both values of e^2 are always negative. These facts have been used in proving the following statements:

For each value of p subject to the condition $0 \leq p^2 < \tfrac{1}{3}\alpha$, two equilateral triangles are found, provided the given distances a, b, c, may be taken as the lengths of the sides of a triangle.

(S) When the conditions $p^2 = \tfrac{1}{3}\alpha$ and $b + c \geq a$ are both satisfied, just one triangle is found. In all other cases no triangle is found.

(D) One of the triangles found degenerates into a point if, and only if, the given distances are equal.

The cases in which the perpendicular from the given point to the triangle plane falls

(1) outside one triangle and inside the other;

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[2] If the given point lies outside of the given triangle we shall have $B + C - A = 0$, but the equations of cosines still hold.—Editors.

(2) on a side of one triangle and outside the other, but (2) (*S*) at a vertex if there is only one solution; and

(3) outside both triangles, also (3) (*S*) outside, if there is only one solution:
correspond respectively to the conditions

$$p^2 \lessgtr \frac{b^2c^2 - (b^2 + c^2 - a^2)^2}{2a^2 - (b^2 + c^2)} = f(a^*bc).$$

The case in which the perpendicular falls

(4) on the side of one triangle produced,
is a special case under case (1) and corresponds to the conditions

$$f(a^*bc) > p^2 = \frac{a^2c^2 - (a^2 + c^2 - b^2)^2}{2b^2 - (a^2 + c^2)} = f(b^*ac).$$

These formulas are indeterminate if, and only if, one of the triangles found is degenerate.

$f(a^*bc)$ and $f(b^*ac)$ are equal only when a and b are equal, and they then reduce to $\frac{1}{4}\alpha$; so that the property (2) (*S*) mentioned above may be regarded as the result of combining any two of the properties (2), (*S*), and (4).

Only the following eight combinations of the above properties are possible,

(1), (1) (4), (1) (*D*), (2), (2) (4) (*S*), (3), (3) (*S*), (*S*) (*D*).

The general conditions which are mentioned above have been derived by "projection" from the corresponding conditions when p is zero, that is, by making the substitutions (1). For example, the conditions that P falls on a side or a side produced are respectively $e = b + c, e = b - c$, which are equivalent to $a^2 = b^2 + bc + c^2, a^2 = b^2 - bc + c^2$. In the latter condition, when the relation, $a \geq b \geq c \geq 0$, is to be preserved, care must be taken to interchange a and b. The value of a which makes P fall on a side produced is then determined by the equation $a^2 - ac + c^2 = b^2$. Substituting a' for a, etc., in this equation gives the corresponding condition for three dimensions, $p^2 = f(b^*ac)$.

III. Historical Notes by R. C. Archibald, Brown University.

This problem has been frequently discussed in books, pamphlets, and periodicals for more than a century. In 1803 L. N. M. Carnot gave indications of synthetic and analytic solutions of the following more general problem (*Géométrie de position*, Paris, 1803, pp. 381–382, 389–390): "Connoisant les trois angles d'un triangle, et les distances de leurs trois sommets à un point donné dans le même plan, trouver les trois côtés de ce triangle." Carnot points out that the method of solving this problem may be applied to solve the following example concerning the pyramid: "Connoissant tous les angles que font deux à deux les six arêtes d'une pyramide triangulaire, et les distances de ses quatre sommets à un point quelconque de l'espace, trouver toutes les dimensions de cette pyramide." Among many discussions of Carnot's problem in the plane, the following may be mentioned: By R. Götting, *Einen Punkt zu bestimmen, dessen Entfernung von drei gegebenen Punkten sich wie drei gegebene gerade Linien verhalten*. Progr. Torgau, 1888. 30 pp. + 1 plate—By Combier, "Note de géométrie," *Journal de mathématiques élémentaires*, vol. 3, 1879, pp. 120–126 (trigonometric discussion)—By A. H. Curtis, M. Jenkins, and J. McDowell, *Mathematical Questions with their Solutions from the 'Educational Times,'* vol. 44, 1886, p. 110; vol. 45, 1886, pp. 68–70.

The particular case for an equilateral triangle, with its vertices on three concentric circles of given radii, was considered in Gabriel Lamé's valuable *Examen des différentes méthodes employées pour résondre les problèmes de géométrie* (Paris, 1818, pp. 81–82), in G. Ritt's *Problèmes d'application de l'algèbre à la géométrie avec les solutions développées* (Paris, 1836, pp. 17–20), and in *Mathematical Questions with their Solutions from the 'Educational Times,'* (*a*) by W. S. McCay and W. S. Burnside, vol. 10, 1868, pp. 98–99; (*b*) by W. J. C. Miller, R. F. Davis, S. A. Renshaw, etc., vol. 26, 1876, pp. 24–28.

In *Nouvelles Annales de Mathématiques*, vol. 3, 1844, p. 376, Prouhet proposed the following problem: "Trois circonférences étant tracées sur un même plan, on propose de trouver sur ces circonférences, *en ne faisant usage que du compas*, trois points qui soient les sommets d'un triangle équilatéral." Solution by Breton (de Champ) is given in vol. 9, 1850, pp. 299–304.

Also solved by R. D. Bohannan, W. F. Cheney, Jr., William Herberg, H. Halperin, H. L. Olson, A. Pelletier, Elijah Swift, and C. C. Yen.

NOTES AND NEWS.

Edited by E. J. MOULTON, Northwestern University, Evanston, Ill.

Assistant professor W. E. MILNE, of the University of Oregon, has been promoted to a full professorship of mathematics.

Mr. F. C. KENT has been promoted to an assistant professorship of mathematics at the Oregon Agricultural College.

In the mathematics department of the University of Minnesota, Professor G. N. BAUER is on leave of absence for the winter quarter, Assistant Professor R. R. SHUMWAY has been appointed to an associate professorship and Mr. R. M. BARTON (*1919*, 420) has been promoted to an assistant professorship of mathematics.

At the University of Michigan, Associate Professor PETER FIELD has been promoted to a full professorship, Assistant Professor J. W. BRADSHAW has been promoted to an associate professorship, and Messrs. E. C. BLANKENSTEIN and O. J. PETERSON have been appointed instructors in mathematics.

Mr. V. G. GROVE, of Cornell University, has been appointed assistant professor of mathematics at Michigan Agricultural College.

Professor L. D. AMES has been made registrar at the University of Missouri.

Dr. H. E. WOLFE (*1920*, 90) has been appointed assistant professor of mathematics at Indiana University.

Dr. TOBIAS DANZIG is an instructor of mathematics in Johns Hopkins University.

Mr. L. H. RICE, of Tufts College, has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

At the University of Maine, Mr. A. S. PRATT, of Brown University, has been appointed instructor in mathematics, and Mr. J. P. BALLANTINE, of Harvard University, instructor in mathematics and physics.

At Teachers College, Columbia University, Professor C. B. UPTON, of the department of mathematics, and secretary of the college since 1911, has been appointed provost.

Professor ALFRED BAKER, forty-four years a teacher of mathematics in the University of Toronto, retired on a Carnegie pension at the end of the academic year 1918–19. Professor A. T. DELURY succeeded him as head of the depart-

ment of mathematics. In the same department Lecturer I. R. POUNDER has been promoted to an assistant professorship.

At the University of Lille, J. CHAZY, professor of general mathematics (*1919*, 371), has been appointed professor of differential and integral calculus in place of the late Professor DEMARTRES (*1920*, 43).

Dr. J. FAIRON has been appointed professor of mathematics at the University of Liège.

Professor W. KILLING, of the University of Münster, has retired from active teaching.

Dr. F. SCHUR recently professor of mathematics at the University of Strassburg has been appointed to a professorship at the University of Breslau in place of the late Professor R. STURM (*1920*, 89); and Professor P. EPSTEIN, also recently at the University of Strassburg, has been appointed privatdozent at the University of Frankfurt a. M.

Dr. H. TIETZE and Dr. G. HESSENBERG have been appointed professors of mathematics at the University of Erlangen.

At the University of Freiburg i. Br., Dr. A. LOEWY has been appointed professor of mathematics, and Professor L. STICKELBERGER has retired from active teaching.

Dr. EMMY NOETHER, daughter of Professor MAX NOETHER, and Dr. SCHMEIDLER have been appointed privatdozents at the University of Göttingen.

Dr. E. CZUBER, professor of mathematics in the Technical High School at Vienna, has retired.

Actuary ARNFINN PALMSTRÖM has been appointed extraordinary professor of the science of insurance at the University of Christiania, with the special duty of conducting the insurance seminar at the university.

Professors E. L. A. MERLIN and L. M. M. STUYVAERT have been promoted to full professorships of mathematics at the University of Ghent.

Associate professor E. GAU has been appointed to the professorship of infinitesimal analysis at the University of Grenoble, succeeding Professor COLLET, who has retired from active teaching.

Professor E. DANIELE, of the University of Catania, has been appointed professor of rational mechanics at the University of Modena. The death is announced of Professor A. RICCO, director of the Observatory of Catania and vice-president of the International Astronomical Union.

Dr. R. C. MACLAURIN, president of the Massachusetts Institute of Technology since 1908, died January 15, 1920 in the fiftieth year of his age. He was born in Scotland and educated in New Zealand and at Cambridge University. He was professor of mathematics at the University of New Zealand 1898–1905, and dean of the faculty of law there from 1905 to 1907, when he was elected professor of mathematical physics in Columbia University.

Professor V. REINA, of the University of Rome, [*1920*, 45] died November 9, 1919, at the age of fifty-three years, and Professor E. MILLOSEWICH, of the Observatory del Collegio Romano, died December 5, 1919, at the age of seventy years. Professor Millosewich was secretary of the R. Accademia dei Lincei.

Professor E. H. BRUNS, director of the Observatory of the University of Leipzig, died September, 1919, at the age of seventy-one; and Dr. O. DANZER, privatdozent in the Technical High School at Vienna died March 26, 1919, as the result of an accident.

The Bordin prize (3000 francs) of the Academy of Sciences of the Institute of France is awarded in mathematics every two years for a memoir on a proposed problem. For 1919 the question was: "In the theory of integrals of total differentials of the third kind and of double integrals relative to an algebraic function of two independent variables, there has been proved the existence of certain integers, whose value is difficult to obtain and may depend on the arithmetic nature of the coefficients of the equation of the surface corresponding to the function. The Academy asks for a detailed study of these numbers in some important special cases." This prize was awarded to Professor SOLOMON LEFSCHETZ, of Kansas University, for a memoir entitled "Sur certains nombres invariants des variétés algébriques avec application aux variétés abéliennes." The principal results of the memoir were given in two notes published in tome 168 of the *Comptes Rendus* of the Academy: "Sur l'analyse situs des variétés algébriques," pages 672–674, March 31; and "Sur les variétés abéliennes," pages 758–761, April 14, 1919. There is a summary of the memoir by Emile Picard in the *Comptes Rendus*, December 22, 1919, pages 1200–1202, being his report for the committee upon the occasion of the formal award of the prize.

In 1882 there was established at Frankfurt a. M. a fund under the name of the Peter-Wilhelm-Müller foundation, with a capital of a million and a half marks, from the income of which, among other things a prize was to be given every third year for the best work in the following domains taken in order: art, poetry and music, philosophy and historical philology, mathematics and the natural sciences. Once this prize was awarded to a physicist by a jury in which a majority were mathematicians, and once it was awarded to a mathematician by a majority of physicists. In 1918 it was again the turn of mathematics and science, and a prize of 10,000 marks was divided equally between Professor D. HILBERT for the totality of his mathematical writings, and Professor A. EIN-

STEIN, for the great mathematical talent displayed in his development of relativity and gravitation theories.

Professor A. G. WEBSTER, of Clark University, has been elected honorary member of the Royal Institution of Great Britain.

Professor W. W. LANDIS, of Dickinson College, has returned to his academic work after serving as a Y. M. C. A. Secretary with the Italian army for over a year. He was accorded the honorary rank of major in the Italian army, and received several medals in recognition of the value of his work.

The mathematical library of the late Dr. R. A. HARRIS has been presented to Cornell University by Mrs. Harris. The collection comprises nearly three thousand volumes, including practically the complete literature on the theory of tides.

At the meeting of the Association of Teachers of Mathematics in New England at Cambridge on January 31, Professor J. L. COOLIDGE gave an address on "The annual meeting of the Mathematical Association of America."

At the meetings of the Chicago and Southwestern Sections of the American Mathematical Society in St. Louis, December 30–31, 1919, forty-four members of the Society were present and twenty-eight papers were read. Professor R. D. CARMICHAEL was elected Chairman, Professor C. N. MOORE member of the program committee, and Professor ARNOLD DRESDEN Secretary of the Chicago Section. This Section will meet in Chicago on April 9 and 10. A feature of the program is a symposium on "The field equations of electromagnetism and relativity" by Professors MAX MASON and A. C. LUNN.

In Section A of the American Association for the Advancement of Science Professor D. R. CURTISS was elected chairman and vice-president for the two years 1920–1921.

The following resolutions were passed by the Council of the American Mathematical Society at the meeting in New York on December 31, 1919:

1. "The Council regards the preparation and publication, in America, of a dictionary of mathematical terms as not only most desirable but also entirely feasible, provided that financial aid for the preparation of the manuscript can be secured.

2. "Impressed with possibilities for the more extensive development of pure and applied mathematics in America, and with the importance of such development to the nation, the Council records its conviction that there are undertakings whose active consideration would be highly desirable if adequate financial assistance might be regarded as available. Among such undertakings are: 1. The preparation and publication by societies or individuals of surveys, introductory

monographs, translations, memoirs, and treatises in important fields, including the history of mathematics; 2. The organization of research fellowships; 3. The preparation and publication of an encyclopædia of mathematics in English; 4. The preparation and publication of an annual critical survey, in English, of the mathematical literature of the world; 5. The preparation and publication of a biographical and bibliographical dictionary of mathematicians."

Nature announces that Girton College, Cambridge, has received a gift of £10,000, the capital and interest of which are to be applied during the next twenty years for the encouragement of scientific research by women in mathematical, physical and natural sciences.

The Mathematics Section of the Texas State Teachers' Association met in Houston, November 28, 1919. There were about one hundred teachers of mathematics in attendance. Dr. GOLDIE P. HORTON, instructor in pure mathematics at the University of Texas, was chairman for the year and presided at the meeting. The following papers were read: "Some extra text-book problems," by Mr. P. H. Underwood, Galveston; "Some definitions of mathematics," by Dr. P. M. Batchelder, University of Texas; "Limit proofs in geometry," by Prof. A. A. Bennett, University of Texas; "Compulsory mathematics in high school," by Miss Helen Carr, Orange; "Some desirable points in elementary texts," by Prof. G. C. Evans, Rice Institute. There was also discussion of these papers, of high school mathematics clubs, and of the work of the National Committee on Mathematical Requirements.

THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS, January 13, 1920 [*1919*, 439–440; 462–463].

The preliminary report, made to the National Committee on Mathematical Requirements by a sub-committee, on "The Reorganization of the First Courses in Secondary Schools Mathematics" and issued as a basis for discussion on November 25, 1919, has received considerable attention during the short period since its publication. It has been discussed at various meetings of teachers organizations, both large and small, and reports on the results of such discussions have been received by the chairman of the National Committee.

In view of the fact, however, that the number of organizations actively coöperating with the National Committee has now reached 54 it is clear that this discussion has only just begun.

The National Committee at its last meeting in New York City, December 30 and 31, 1919, subjected the sub-committee report to a careful revision in the light of the suggestions, criticisms and comments received. The revised version of the report was published in February by the U. S. Bureau of Education (*Secondary School Circular*, No. 5, 12 pp.) as a preliminary report by the National Committee. *Comments, criticisms, suggestions for improvement, etc., are earnestly solicited from organizations, committees, and individuals.* They should be sent to the Chairman of the Committee (J. W. Young, Hanover, N. H.).

The response to such requests in the past has been most gratifying, and augurs well for the future. Only through coöperation on a large scale on the part of all interested in the problem of improving the teaching of mathematics can success be assured.

The mailing list to be furnished the U. S. Bureau of Education for the Reports of the National Committee is now in preparation. This list now includes all the members of the American Mathematical Society, the Mathematical Association of America, the Association of Teachers of Mathematics in New England, the Association of Teachers of Mathematics in the Middle States and Maryland, and the teachers of mathematics who are members of the Central Association of Science and Mathematics Teachers and many others. Requests have been sent to all the organizations coöperating with the National Committee for lists of their members. *Individuals desiring to be placed on the mailing list of the National Committee should send their names and addresses to the Chairman of the Committee without delay.*

A preliminary report on the principles to govern a revision of College Entrance Requirements was submitted by the National Committee to the Councils of the American Mathematical Society and of the Mathematical Association of America on December 30, 1919.

A report whose early publication may be expected is "The Doctrine of Formal Discipline and the Transfer of Training" (a critical examination of the literature by Vevia Blair).

Professor R. C. Archibald of Brown University has consented to prepare a report for the National Committee on the desirable professional training for teachers of mathematics in the United States and on courses primarily intended for prospective teachers.

The preparation of reports on Sources and Desirable Types of Problems, and on Elective Courses in Mathematics for Secondary Schools (their aims, content, organization, etc.) has begun, and exhaustive and authoritative investigations of the mathematical elements entering into the work of the various vocations, industries, and professions, are under consideration.

The appropriation of $16,000 by the General Education Board under which the National Committee is operating, was made to cover the expenses of the Committee for one year ending July 1, 1920. At the last meeting of the Committee, it was voted to request the General Education Board to give the Committee its support for another year ending July 1, 1921.

The Committee heartily approves the proposed organization of a National Council of Mathematics Teachers and has authorized its officers to attend the organization meeting of the Council in Cleveland on February 24th in connection with the meeting of the Department of Superintendence of the National Education Association.

The date and place of the next meeting of the National Committee have been tentatively set for Chicago, Illinois, on April 23rd and 24th, 1920.

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